India,[1] but this would not be at all necessary. He was as careful a writer as any of his time, a scientist of high repute, a student of the history of sciences, and a man less given to the acceptance of mere tradition than was usually the case. It is to be expected from his reputation that he would have consulted the best Arabic sources available, although we have at present only slight knowledge of what those sources were, and although no manuscripts thus far translated throw any light upon the problem.

The following is a translation of the introduction written by Rabbi ben Ezra:

"In the name of the Most Holy and Revered, in whose help I trust, spake Abraham ibn Ezra the Spaniard. In ancient times there was no wisdom and no [true] religion among the sons of Ishmael, the tent dwellers, until the [author of the] Kora[2] came and gave to them from his heart a new religion.

"After him there appeared many sages among them, who wrote many books on their laws; but at last there appeared a great king in Ishmael, called e's Saffah,[3] who heard that there were many sciences in India. And he gave orders to search for a scholar who should know the language of India and that of Arabia, so as to translate for him one of their books of wisdom, although he feared that a calamity might befall him,[4] since profane sciences [were then permitted] in Ishmael in the book of the Koran alone, and whatever of the sciences they received [by tradition] was [believed to be] therein. [He had heard that] in India there was a book, very important in the councils of the kingdom, that was arranged in the form of stories put in the mouths of dumb creatures, the large number of pictures rendering the book very valuable in the eyes of the reader. And the name of the book was Kalilah we-Dimnah,[5] which means the Lion and the Bull, because the first gate[6] of the book refers to them.[7] And the above named king fasted forty days, hoping to see the angel of dreams who should

[1] Steinschneider adduces a proof that Ibn Ezra did not visit India. See *Zeitschrift der deutschen Morgenländischen Gesellschaft*, Bd. XX, p. 430.

[2] For *koran*.

[3] In the De Rossi codex this name appears as Altsaphak; but evidently the Abbasside el-Saffah is meant, and this fixes the time as the middle or second half of the eighth century. A similar story is told of Nushirvan the Just, who reigned in Persia in the sixth century.

[4] That is, the translator. This calamity might befall one who assumed, by translating such a work, that the Koran was not all sufficient. In the Hebrew a few words are omitted but this seems to be the sense of the passage.

[5] Called by the Hindus the Kurtuk Dumnik.

[6] That is, the first chapter.

[7] Although the first chapter (in some editions the second) of the Kalilah we-Dimnah deals with the bull and the lion, the name of the book is derived from the names of two jackals, Karattaka and Damanaka, which play an important part in the stories of the subsequent chapters. See Theodor Benfey's introduction to G. Bickel's edition of the Syriac *Kalilag und Damnag* (Leipzig, 1876, page 11); Wollaston's English translation under the title *Lights of Canopus* (London, 1904); J. Derenbourg, "Kalîlâh et Dimnâh" in the *Bibliothèque de l'école des hautes études* (Paris, 1881). The first edition appeared in Johannes de Capua's *Directorium humanae vitae*, "Et vocatur liber Kelile et dimne," Strassburg, c. 1488–1493.

In the Talmud, mention is made of Ben Tiglah and Ben Laanah, and the relation of these to the Kalilah we-Dimnah was shown by Wolf, in the *Bibliotheca Hebraea* I, 932, note. See also Steinschneider, in *Jewish Literature*, London, 1857 (p. 279, note 54, *a*). Geiger, in the *Jüd. Zeitschrift*, VII (1869), 139, sets forth his belief that Ben Tiglah and Ben Laanah are the names of two Jewish authors.

allow him to translate the book into Arabic. Then he had a dream in harmony with his thoughts.[1] He thereupon sent for a Jew who lived in his time and who knew the two languages, and he gave him command to translate the book, since he feared that if an Arab should translate it he might die.[2] And when he saw how wonderful the book was, and so it really is, he was overcome by a desire to know more. Then he gave great wealth to the Jew so that he might journey to the city of Arin[3] on the equator, under the signs of the Ram and the Scales, where the day throughout the year is equal to the night, [thinking '] Perhaps he will succeed to bring one of their wise men to the king ']. And the Jew went [there] and indulged in many subterfuges, after which, for a large sum, one of the wise men of Arin agreed to go to the king, and the Jew swore to him that he would not detain him beyond a year and that he would return him to his home. Then this scholar, whose name was KNKH,[4] [was taken to the king] and taught the Arabs the bases of number, that is, the nine numerals.[5]

"From the mouth of the learned man, through the Jew [as] an Ishmaelite interpreter, a scholar by the name of Jacob ibn Sharah[6] translated the book of the tables of the seven planets and the creation[7] of the earth,[8] the degree of rise, the establishing of the houses,[9] and the knowledge of the upper stars and the darkening of the lights.[10] No explanation of these matters was set forth in the book, only operations in the form of rules to be accepted on faith.[11] The average motion of the planets was computed according to Hindu methods, their cycle, called Hazervan, being equal to 432,000 years.[12]

---

[1] This seems to have been looked upon as a sanction for the translation of profane works, for el Mâmûn waited until he saw Aristotle in a dream before ordering the translation of the Greek philosophers.

[2] That is, an Arab might be punished for the profanity of such a translation while a Jew might escape.

[3] Arin seems to have been an astronomical center of India, like Ujjain. Indeed, Reinaud thinks the two places were the same. Al Khowarizmi's tables were constructed on the basis of this meridian. See Suter's edition of the tables, *l. c.*

[4] In the Hebrew text the vowel points are not given. The name may be Kanka or Kanaka; for, according to Meǵriti (959), Kanka was the inventor of amicable numbers.

The subject is deserving of a more extensive report than is possible in this connection.

[5] Evidently the decimal system, else "bases of number" would have no significance.

[6] Since in the Spanish rabbinical script the Hebrew letters *shin* and *theth* are easily confused, as also the letters *he* and *qaf*, a scribe who was not familiar with the name might easily have written Sharah for Tarik. Hence this Jacob ibn Sharah may have been the same as the famous astronomer and astrologer Ja'qûb ibn Târiq mentioned by el-Bîrûnî as having been living in 777.

[7] Literally, "the whole work."

[8] Was this the Sûrya Siddhânta, the great astronomical work of the Hindus, written in the fourth century?

[9] That is, in the scheme of astrology. The passage is very obscure, and several words omitted at this point in this translation seem to have referred to the ascension and declination of the planets. Steinschneider was also unable to find the exact meaning of the passage.

[10] Eclipses of the sun and moon.

[11] The Hindus gave no proofs of their propositions, as may be seen by examining the works of Âryabhata, Brahmagupta, Mahâ îr, and Bhaskara, all of which are now accessible in translation.

[12] In the Sûrya Siddhânta is this passage: "To determine the saura years contained in this aggregate, write down the following numbers, 4, 3, 2, which multiply by 10,000; the product, 4,320,000, is the aggregate or Mahâ yuga. . . . Divide the aggregate 4,320,000 by 10 and multiply the quotient by 4 for the satya yug." See *Asiatic Researches*, 1790, II, p. 230.

"After [the death of] Jacob the translator there arose a great scholar in Ishmael who knew the secrets of the wisdom of counting [and of] chronology and [who] reduced the average planetary motion to the era of Ezdeger,[1] the last of the Persian kings, for the Arabs conquered the kingdom of Persia and converted the inhabitants to their own religion. This scholar was Mohammed ibn Mûsâ al Khowârezmî,[2] and all later Arabic scholars do their multiplications, divisions, and extraction of roots as is written in the book of the [Hindu] scholar which they possess in translation.[3] He[4] prepared, in a more convenient form for students, tables which were the equal of the work of KNKH, but he gave no explanation of his statements. After him there arose in Ishmael[5] a scholar called el-Ferġânî,[6] who set forth reasons for the words of al Khowarizmi. After el-Ferġânî[7] another great scholar translated a very important book about the stars, written by Ptolemy, king of Egypt,[8] a Greek who lived a thousand years ago.[9] This book[10] is perfect, there being nothing higher in the science of the spheres, their secrets, their distances from the earth,[11] and the measure of the upper stars in the sphere of the zodiac. He divided them[12] into six classes, of which the first [class] was called the first glory. He established the number of stars in each class and enumerated all [the stars] in the forty-eight constellations,[13] namely, the constellations of the entire sphere which contains 1,022 stars besides the clouded ones.[14] He gave reasons for all corrections[15] and, in general,

It is interesting to note that the number 432,000 also appears in the Babylonian chronology. According to Berossus, a Babylonian priest (250 B. C.), the antediluvian kings of Babylon reigned for a total period of 432,000 years. See W. H. Roscher, *Die Zahl 40 im Glauben, Brauch und Schrifttum der Semiten*, Leipzig, 1909, p. 97; H. V. Hilprecht, *Mathematical . . . Tablets from . . . Nippur*, Philadelphia, 1906, p. 21.

[1] This begins with the day when Jezdegird III ascended the throne of the Sassanides, June 16, 632, of the Christian calendar of that period. Although the Mohammedan conquest soon imposed the calendar of Islam on most of Persia, the Jezdegirdian calendar survived among the Persians in the southern provinces and in western India.

[2] Mohammed the son of Moses, the Kharezmite, who flourished in Bagdad early in the ninth century and whose *Algebr w'al-Muqabala* gave the name to algebra.

[3] It was probably this work which al Khowârizmî made the basis of the treatise on computation which, as is well known, was translated into Latin in the twelfth century under the title *Liber Algorismi* (Book of al Khowârizmî), from which we have the word *algorism*.

[4] Al Khowârizmî.

[5] In the original, Israel, probably a mistake of the scribe.

[6] Ahmed b. Muh. b. Ketîr el-Ferġânî or el-Ferġûnî was one of the astronomers and followers of el-Mâmûn.

[7] In the original, "after him."

[8] In the Middle Ages, and even later, Claude Ptolemy (second century A. D.), the great astronomer, was commonly mistaken for one of the kings who reigned under the name of Ptolemy.

[9] This was approximately correct, since Ibn Ezra lived in the twelfth century, while Ptolemy lived in the second century.

[10] The Almagest.

[11] The original may also be translated as motion with respect to the earth.

[12] The stars.

[13] Literally, "figures."

[14] Ptolemy's tables, edited by Peters and Knobel (Washington, 1915), give 48 groups or constellations and 1028 stars. The number 1,022 is not only given by Ibn Ezra but is also given by el-Ferġânî in his compendium of the Almagest, as shown in a Hebrew MS. of the latter's compendium now in the Columbia University Library (X 893, T 522). This shows that the number in Ibn Ezra's work was not an error of some scribe. Moreover, in the Manitius edition (1912) of the Almagest the number is given as 1,022.

for everything found in the tables translated by Jacob from the mouth of the Indian scholar.

"All the proofs given by Ptolemy or Talmi in his great book Almagest are perfect, and no man can contradict them, for these are proofs from the science of magnitude or the science of measurements, which is called in the Greek tongue Yeometria and, by the holy sages of Israel, Gematria.[1]

"On this book[2] commentaries were written by many sages in Ishmael, the most distinguished among them, in mathematics and astronomy, being the scholar Muhammed ibn Mutani.[3] He [it was who] compiled for one of his family a very important book about the corrections[4] of the planets and the explanation of the contents of the tables in the book of al Khowarizmi, and briefly mentioned the proofs and illustrations, their principles being taken from the book [called] Almagest. In certain passages, however, his explanations are more complete than those of King Ptolemy, and in these are also places where he sets forth mathematical proofs [which were] invented by himself.[5] In most places he agrees with the theories of el-Ferġânî who was mentioned above. For the sake of clearness his book is arranged in the form of questions and answers.

"Said Abraham [ibn Ezra]: Except in a few places there is no difference between the norms[6] of the planets as given in Ptolemy's work and those of the Hindu scholar, and at the proper time I shall mention how this difference arises. I have written a book on the mean motion of the planets, and on the head and tail[7] of the *tali*[8] as observed by the astrolabe, because the positions of the planets

[15] Steinschneider translates the Hebrew word *tikunim* as "norms," but the word also means corrections or interpolations, and this is much more in harmony with the context at this point, although in two later sentences the word "norms" seems more appropriate.

[1] This is not correct, for gematria was an entirely different science, in no way related to geometry. It is probable that the error is that of some copyist of Ibn Ezra's manuscript, although the latter may himself have been at fault. Without the vowel points the word can be read either geometria or gematria. Because of this fact the pronunciation of the word as it appears in the Talmud was quite unsettled until the nineteenth century. Ibn Ezra asserts elsewhere that the Almagest was translated into the Arabic after the time of el-Ferġânî, and at any rate after the time of al Khowarizmi. This confirms the view, expressed by Weber (*Naxatra*, I, 321), that the Arabs were familiar with the science of the Hindus before they became acquainted with the works of Ptolemy. (See also Woepcke, *Sur l'introduction de l'Arithmétique indienne en occident*, Rome, 1859, p. 58, and Lassen, *Indische Alterthumskunde*, II, 1139.) Reinaud (*Abulfeda*, pp. XLI–XLIII) states that the Almagest was translated completely into Arabic under el-Mâmûn, but that it was translated into Syriac and Hebrew in the middle of the eighth century, a doubtful statement as to the Hebrews for the reason that they were not generally interested in such matters at that time. See Steinschneider in the *Zeitschrift der Deutsch. Morgenl. Gesellsch.*, XXIV, page 337.

[2] The Almagest.

[3] The Muhammed ibn Ahmed el-Bîrûnî referred to in the note on page 100.

[4] See the note on the Hebrew *tikunim*, *supra*.

[5] Literally, from his heart.

[6] See the note on the word *tikunim*, *supra*. We have taken Steinschneider's translation here, since it seems to make better sense than "corrections." What is apparently meant is "tables."

[7] In the Hindu writings, Rahu and Ketu.

[8] The *tali* was a celestial dragon believed by early writers to be the cause of eclipses and of various other disturbances in the heavens. The Chinese attributed eclipses of the moon to the fact that it was covered by the head or the tail of the dragon, and to prevent it from being devoured they tried to frighten the dragon away by the noise of cymbals and tambourines.

The word *atalû*, used by the Babylonians to mean eclipse, is evidently the source of the

in the tables of al Khowarizmi do not agree with their observed positions by 9½ degrees. It is my opinion that the idea of the Hindu scholars as to mean planetary motion is based upon the representation on the plane, and this is correct according to the science of projection but not according to the science of astronomy.[1]

"The tables in Almagest are useless for the reason that they are evidently corrupted. Moreover, they are not in accord with the paths of the stars.[2] The errors are not due to Ptolemy, however, but to the ancients from whom he derived [his information], and this I shall discuss later when I shall have completed this work. The norms in my book are the same as Ptolemy's and as those used by all Arabian scholars. [The latter] prepared many tables and were more exact in their work than Ptolemy, and I shall hereafter explain the reason. Only the norm of the sun is not[3] the same as Ptolemy's, being less than that by 29 minutes, for he relied upon the observations of ABRKS[4] who lived 208 years before Ptolemy.[5] He could not have relied upon the testimony of Fitin and Afitimon[6] who lived about 1,000 years before him,[7] for they could not have made [as good] an astrolabe [as the one] used by Ptolemy. Hipparchus had stated that the position of the sun at its apogee[8] was in his time at 5 degrees of the Twins,[9] and since Ptolemy found it in nearly the same place he inferred that the position of apogee of the sun, unlike the positions of apogee of the five planets, does not change; but many scholars found that it changes as well as

---

Syriac word *atalia* and the Hebrew *tali*, both of which signify dragon and are used in connection with eclipses.

This idea of the power of the *atalia* was ridiculed by Severus Sebokht in the seventh century, and of course Ibn Ezra had no illusions concerning any dragon.

See Morris Jastrow, *Religious Belief in Babylonia and Syria*, page 213; M. F. Nau in the *Journal asiatique*, 1910, ser. X, Vol. 16, page 219.

[1] Literally, "is according to the image of the galgal," etc. Galgal means a circle, and the passage probably means that the Hindus studied planetary motion from the representation of the heavens as circular, on a plane, rather than from a celestial sphere. The passage is more obscure than any other in the book, and Steinschneider was also unable to satisfy himself as to its exact meaning.

[2] Literally, "do not follow the way of the images." The passage is obscure, and may refer to the paths of the planets in the constellations or to the zodiac.

[3] In Ibn Ezra's work here described.

[4] That is, Hipparchus, who lived about 140 B. C. Query: is there any relation of Abrks to the Gnostic term abraxas?

Peters and Knobel, *l. c.*, page 7, agree with Delambre that the catalogue of stars in the Almagest is due to Hipparchus. This was Ibn Ezra's opinion.

[5] This is only approximately correct, since Ptolemy lived in the second century A. D. Probably the 208 is a scribe's error for 280, since Ptolemy himself speaks of Hipparchus as having lived 285 years earlier than he.

[6] That is, Meton and Euctemon, who flourished in the fifth century B. C. Ptolemy states (*Almagest*, III, 1, which is Vol. I, p. 141 in the translation of Manitius) that he had to omit all reference to the observations made by the school of Meton and Euctemon. On the names used by Ibn Ezra, see Isak Israel, Jesod Olam, IV, 7; *Mag. für die Literatur des Auslandes*, 1846, p. 378.

In the fragment of Levi ben Abraham's work on astrology (Cod. Reggio, 13, fol. 56, now Cod. Oxf. 2028) the names appear as Meton and Euctemon.

[7] Probably 600 years would be nearer the truth

[8] Literally, "Place of the high point of the sun."

[9] He found it to be 5° 30′ in the year 140 B. C.

in the case of the planets. Its position now, in the year 1160 of the era of the uncircumcised,[1] is 25 degrees of the Twins.

"And these are the learned men of Ishmael who observed the point of apogee of the sun, not all of whom lived in the same generation:

"The first was the Arabian scholar[2] . . . and Jahjâ ibn Abî Manṣûr[3] and el Merwadi[4] and Ibn al Mokaffa-a,[5] el Kufi,[6] Jacob al Kindi,[7] Thabit ibn Qora,[8] al Hakemi[9] the Hindu, Theon of Alexandria,[10] Ibrahim ez-Zarkali[11] the Spaniard, el-Batani,[12] Ibn Alostay,[13] and Ibn el-Alam.[14]

"And now I shall begin to translate the book of the Ishmael scholar.[15]

"Here beginneth the book of Muhammed ibn al Matani ibn Abdul Karasi[16] ibn Ali Ishmael explaining[17] the tables of al Khowarizmi. You remember what you saw in [the] tables of the planets, [namely,] mistakes and disagreements and [evidences] that their authors did not give any proof for what they told us to do, but they left it[18] to us, and presented them[19] as a matter of tradition without [any] discussion. In the case of a book of this kind the reader may attribute to its

---

[1] The date of the work, seven years before Ibn Ezra's death.

[2] The name is omitted in the Hebrew text.

[3] He lived in the time of the caliph el-Mansûr. See the *Zeitschrift für Mathematik*, XII, 31, seq.; Suter's list, page 8.

[4] Steinschneider identifies him as possibly Merwesi or el Merwadsi, the Habas el Hâsib el-Merwazî of Suter's list. He lived at Bagdad in the time of el-Mâmûn.

[5] Not in Suter's list.

[6] Probably an error for el-Sûfî, that is, 'Abderrahmân ibn 'Omar, Abû'l-Hosein, el-Sûfî, who died in 986. His work on the fixed stars was translated into French in 1874. Possibly, however, Ibn Ezra means Muhammed ibn Zijâd ibn el-A'râbî of Kûfa, who wrote on astronomy as well as language, and who died about 846.

[7] For the translation of his works into Hebrew, see *Zeitschrift der Morgenl. Gesellsch.*, XVIII, 131, 181. For his work on Hindu arithmetic, see Woepcke, *Mém. sur la propagation*, 159.

[8] Thâbit ibn Qorra ibn Merwân, Abû'l-Hasan, el-Harrânî (826–901), one of the foremost Arab astronomers.

[9] Queried by Steinschneider. There were several Arab scholars of this name, but doubtless the caliph Hakem (996–1021) is meant, after whom the Hakem Tables composed by Ibn Junis were named. The words "the Hindu" are manifestly incorrect.

[10] Of course Theon was not one of the "learned men of Ishmael," so that Ibn Ezra uses the term rather loosely.

[11] Ibrâhîm el Zarquâla, or Zarqâlî, a famous Spanish instrument maker of the eleventh century. See *Zeitschrift für Math.*, XII, 34, 36; Steinschneider, *Etudes sur Zarkali*, Rome, 1884.

[12] Muhammed ibn Gâbir ibn Sinân, Abû Abdallâh, el-Battânî, a famous astronomer, known in Europe under the name Albatagnius. He died in 929.

[13] The name is doubtful.

[14] Probably the astronomer and astrologer 'Alî ibn el-Hosein, Abû'l Qâsim el-'Alawî, known under the name Ibn el-A'lam who died in 985.

[15] Beginning at this point, Steinschneider published only the Hebrew text of this fragment from Bodleian MS., Michael 400, without translation or comment. The rest of the text is still unpublished. See also the *Zeitung d. Morgenl. Gesellsch.*, XXV, p. 421.

[16] The transliteration is doubtful, Steinschneider gives Al Karuz. On el-Bîrûnî see the note on page 100.

The Codex Mich. 835 begins: "This book was written by Ahmed . . . elkerim for his brother Muhammed ben Ali ben Ishmael" etc.

[17] Literally, "about the reasons of."

[18] That is, the proving of the operations.

[19] That is, the rules.

author one of two things: either he does not himself understand the explanations, having merely learned the facts from someone, or he is jealous of his great wisdom and does not care to reveal it. We have already seen that scholars of undoubted wisdom did the same, as in the case of Al Khapash[1] in his work on grammar, known as Aloust, the result being that men learned in the science of grammar decided that his book was suited neither to a teacher nor to a student. You recall that you have found the same thing in al-Khowarizmi's tables and you attributed it to the condensed form[2] of the work or to the selfishness of the author.[3] You also remember that you found the work attributed to el-Ferġânî very far from perfect and even unsatisfactory for the uses to which you might wish to put it, and you saw that he explained things that are clear and are easy of comprehension while omitting all that was difficult and complicated, and you asked me for explanations so that nothing should be concealed from you. Therefore I explained all you asked for, and this will help your understanding and will satisfy any mathematician and man of science like yourself, and may God be my helper.

"Referring to what you have said about el-Ferġânî, [namely,] that you found his work far from perfect,—I have read it and have also found it so; but I have found in el-Ferġânî's book many things that showed him to be a wise man;[4] and the thought has occurred to me that el-Ferġânî had worked out in his mind the commentary on the book and the proofs [to be furnished], but death overtook him and he was unable to complete it. Now, however, someone has transcribed it and attributed to him [as complete] what still lacks the explanations. It is also possible that he completed his work before he died, but that part of it was lost, or that the book fell into the hands of some ignorant person who ruined its perfection, and so we cannot attribute these faults[5] to a lack of knowledge.[6] It is true that these things will not be understood, however, by a man who is ignorant of mathematics.[7] I have already written a book on selected topics of this work and have arranged it in gates[8] in the form of questions and answers, so that it explains more fully all that you may wish to know, and this is easier to understand and more convenient to study and remember, and I trust in the Creator that I may succeed in satisfying your wish. Question: Why did Muhammed al Khowarizmi say that to compute the Arabian month [it is necessary] to take the whole number of the Arabian year, write it in two different places, multiply one [number] by 354, retaining it; then multiplying the other by 13, dividing [the product] by 30, and then adding the quotient to the other result?"

This completes the translation of the Hebrew text as published by Stein-

[1] The name is doubtful. It is probably Akhfash, who died 830–835.
[2] Literally, shortness or brevity.
[3] That is, his jealousy lest others should share his wisdom.
[4] That is, that he was right.
[5] Literally, "things."
[6] That is, on the part of el-Ferġânî.
[7] That is, by such a man as evidently transcribed the work.
[8] *I. e.*, in chapters.

schneider from the Parma codex, and it covers Ibn Ezra's introduction. The text of el-Bîrûnî is in a manuscript in the Bodleian Library, and has never been published.

THE THEORY OF SIMILAR FIGURES.

By ROGER A. JOHNSON, Western Reserve University.

The object of this paper is to place in a new and perhaps a more satisfactory light the elements of the well-known theory of two and of three directly similar figures in a plane. The attempt has been made to keep in the foreground the idea of determining each of such a set of figures by a base-line, that is, by one of a set of homologous line-segments. This notion clarifies the conception, and appreciably simplifies the treatment; further, as shown below, it enables us to avoid certain false proofs which are to be found in the usual discussions of this subject. The use of directed angles[1] is again advantageous, but of course not essential.

The fundamental operations of elementary plane geometry are four in number:

(*a*) *Translation*, or motion of a figure such that every point is moved the same distance in the same direction,

(*b*) *Rotation* about a fixed point,

(*c*) *Reflection* with regard to a fixed line, which is the same in effect as turning the plane over on this line as axis,

(*d*) *Expansion* with regard to a fixed point, whereby the distance of each point of the figure from the fixed point is increased or decreased in a constant ratio (the same word *expansion* is used in all cases, whether the figure is actually enlarged or diminished).

If any figure is subjected to any succession of these operations, the resulting figure is similar to the original one; conversely it is easy to see that if two similar figures lie in a plane, one can be brought to coincide with the other by a combination, for instance, of a translation, a rotation, and an expansion. Our first object shall be to simplify this combination.

As a first lemma, we may note that if we operate on one of two directly similar figures in such a way that two of its points come to coincide with the corresponding points of the other, the figures coincide throughout.

Temporarily we shall use the word *homology* to designate the combination of a rotation and an expansion with regard to the same center.

THEOREM 1. *Given two line segments MN, $M'N'$, there exists a single operation, either a translation or a homology, that carries MN into $M'N'$.*

Case i. If MN and $M'N'$ are equal and parallel, and extend in the same direction, a translation that carries M into M' will carry N into N'.

Case ii. If $MNN'M'$ is a trapezoid, and MM' meets NN' at P, an expansion

[1] "Directed Angles in Elementary Geometry," R. A. Johnson, this MONTHLY, March, 1917, p. 101 ff.

with regard to P, followed by a rotation through 180° when P is between the parallels MN and $M'N'$, carries M and N into M' and N' respectively.

Case iii. If MN and $M'N'$ are segments of the same straight line not coming under case i, we may take a segment PQ, so that $M'N'QP$ is a rectangle. Let R be the point located as in case ii for MN and PQ, and let C be the foot of the perpendicular from R to the line $MNM'N'$. Then MN is carried into $M'N'$ by an expansion with regard to C, followed if necessary by a rotation of 180° about C.

Case iv. In the general situation, MN and $M'N'$ intersect at a point A. Let circles $MM'A$ and $NN'A$ intersect a second time at C; then triangles MNC and $M'N'C$ are directly similar, and if we rotate $M'N'C$ about C until CM' falls on CM, then expand until M' falls on M, we shall have moved N' to the position N.

In particular, one or two of the given points may fall at A: for instance, let A and M coincide; then for the circle $MM'A$ of the proof just given, we use the circle through M and M' which touches MN at M, and the proof holds as before. If both M and M' are at A, or if the circles are tangent there, then the point C is also there.

THEOREM 2. *If two directly similar figures lie in a plane,*

(*a*) *If they are congruent, with corresponding sides parallel and extending in the same direction, a translation will carry one into coincidence with the other.*

(*b*) *In every other case, there exist a point C, called the center of similitude, a number, the ratio of similitude, and an angle, the angle of similitude; if one figure is rotated about C through this angle, and expanded from C in this ratio, it is thereby made to coincide with the other figure.*

COROLLARY. The center of similitude is self-homologous in the two figures, and is the only self-homologous point; in other words, if we can determine a self-homologous point, we thereby find the center of similitude.

We notice a few interesting applications.

THEOREM 3. *If one vertex of a triangle of constant form is held fast, while a second vertex traces any figure whatever, the third vertex traces a similar figure.*

THEOREM 4. *If one of two directly similar figures is held motionless, while the other rotates about a fixed point, the center of similitude traces a straight line or a circle, according as the given figures are equal or unequal.*

COROLLARY. Obviously two circles may be regarded as similar figures in an infinite number of ways, with any point of the one homologous to any point of the other; the locus of the centers of similitude is a circle, the locus of a point whose distances from the centers of the given circles are proportional to the radii of the latter. This is called the *circle of similitude* of the two circles.

THEOREM 5. *If a figure is moved in a plane without change of form or size, so that two of its points remain constantly on fixed intersecting lines of the plane, the locus of the "instantaneous center of rotation" is a circle whose center is the intersection of the fixed lines.*

We may interpret these results in terms of maps of a plane region superposed,

the same side up, on a table. We see that there is, except in one special case, one point whose images in the two maps coincide; this may be called the *double point* of the two maps. In practise this double point would of course often fall outside the boundaries of actual maps. If one map is held fast, while the other is rotated about a pivot, the double point traces a circle or a straight line, according as the maps are on different scales or the same scale. Our other theorems can as well be interpreted in this way.

We consider next the properties of three maps lying on a table; in other words, the relations among three directly similar figures in a plane. Each two of them determine a double point, or center of similitude. We will designate the figures by I, II, III; then the center of similitude of II and III shall be called C_1, that of III and I, C_2, and that of I and II, C_3. Thus C_1 is self-homologous in II and III; and so on.

It is immediately evident that the situation admits numerous special cases, for even in the simple case of two figures (Theorem 1) there are several cases to consider. We shall confine ourselves to the most general case; similar methods may be applied to any special case, yielding similar results. For instance, we shall exclude case *i* of Theorem 1; thus we shall always have three actual centers of similitude.

Let us then consider three similar figures, determined by homologous segments or base-lines, M_1N_1, M_2N_2, M_3N_3; let these lines be not concurrent nor parallel, but form a triangle $L_1L_2L_3$. If C_1, C_2, C_3 are the centers of similitude, triangles $M_2N_2C_1$ and $M_3N_3C_1$ are similar; and so on. We know that C_1 is the intersection of circles $L_1M_2M_3$ and $L_1N_2N_3$; and so on. Now the circles $L_1M_2M_3$, $L_2M_3M_1$, $L_3M_1M_2$ intersect at a point M, and the circles $L_1N_2N_3$, $L_2N_3N_1$, $L_3N_1N_2$ at a point N.[1]

THEOREM 6. *The points M and N lie on the circle or line* $C_1C_2C_3$.

For

$$\begin{aligned} \measuredangle\, C_2MC_3 &= \measuredangle\, C_2M,\ L_2C_2 + \measuredangle\, L_2C_2,\ L_3C_3 + \measuredangle\, L_3C_3,\ MC_3 \\ &= \measuredangle\, MC_2L_2 + \measuredangle\, L_3C_3M + \measuredangle\, L_2C_2,\ L_3C_3 \\ &= \measuredangle\, L_2C_2,\ L_3C_3. \end{aligned}$$

Similarly

$$\measuredangle\, C_2NC_3 = \measuredangle\, L_2C_2,\ L_3C_3.$$

If now L_2C_2 and L_3C_3 are not parallel, but meet at a point X, then the circle through C_3, M and N passes also through C_2, and therefore through C_1. X is also on this circle.

On the other hand, if L_2C_2 and L_3C_3 are parallel, the above equations show that M and N lie on the line C_2C_3, and it comes at once that C_1 is on the same line, and L_1C_1 is parallel to L_2C_2 and L_3C_3.

COROLLARY. We may construct a system wherein the centers of similitude are collinear, or a system wherein they lie on an actual circle.

[1] *Loc. cit.*, p. 103.

For if the triangle $L_1L_2L_3$ and segments M_2N_2 and M_3N_3 are given, C_1 can be found. We may put C_2 anywhere, then C_3, M_1 and N_1 can be located. According as we make C_1L_1 and C_2L_2 parallel or intersecting, $C_1C_2C_3$ will be a line or a finite circle.

In the sequel we shall confine our attention to the case that C_1, C_2, C_3 lie on a circle of finite radius; the reader can easily deduce the corresponding theorems when they are collinear. The circle $C_1C_2C_3$ is called the *circle of similitude* of the three figures, and the triangle $C_1C_2C_3$ is called the *triangle of similitude.*

THEOREM 7. *The lines L_1C_1, L_2C_2, L_3C_3 are concurrent at a point X on the circle of similitude. That is, the triangle of any three homologous lines is in perspective with the triangle of similitude; the center of perspective lies on the circle of similitude.*[1]

For we have just seen that L_2C_2 and L_3C_3 meet at a point X on the circle of similitude; each of these must also meet L_1C_1 on the same circle. But L_3C_3, for instance, meets the circle at C_3 and at only one other point; in general, L_1C_1 and L_2C_2 do not pass through C_3; hence all three of these lines are concurrent at X.

THEOREM 8. *If the lines XL_1, XL_2, XL_3 are divided proportionally at L_1', L_2', L_3'; then $L_2'L_3'$, $L_3'L_1'$, $L_1'L_2'$ are parallel respectively to L_2L_3, L_3L_1, L_1L_2, and are homologous lines in the similar figures. In particular, lines through X and parallel to L_2L_3, L_3L_1, L_1L_2 respectively constitute a set of concurrent homologous lines.*

THEOREM 9. *Conversely, if three homologous lines are concurrent, the point of concurrence lies on the circle of similitude.*

The easiest proof is made by returning to first principles. Let M_1N_1, M_2N_2, M_3N_3 be homologous segments whose lines are concurrent at Z_1. Let an inversion with regard to Z as center carry M_1, N_1, etc., into M_1', N_1', etc. We now have two triangles $M_1'M_2'M_3'$, $N_1'N_2'N_3'$ in perspective at Z; therefore the intersections of corresponding sides are collinear. Let $M_2'M_3'$ meet $N_2'N_3'$ at C_1'; then inverting back to the original figure, the inverse of this point C_1' is the intersection of circles M_2M_3Z and N_2N_3Z, or in other words the center of similitude C_1. But if C_1', C_2', C_3' lie on a line, then C_1, C_2, C_3 lie on a circle through the center of inversion Z, and the proof is complete.

THEOREM 10. *There are three fixed points on the circle of similitude; any set of concurrent homologous lines pass respectively through these points. They are called the invariable points of the system.*

---

[1] So far as the writer can ascertain, this well-known theorem has never been proved. The proofs, as given by McClelland and Lachlan, for instance, are based on a false theorem, namely: "if two triangles are $A_1A_2A_3$ and PQR; if the lengths of the perpendiculars from P to A_2A_3, A_3A_1, A_1A_2 are p_1, p_2, p_3, and so on; then if A_1P, A_2Q, A_3R are concurrent,

$$\frac{p_2q_3r_1}{p_3r_1q_2} = 1$$

and conversely." The converse, which is used in our theorem, is false: for example, let P and Q lie on the bisectors of the interior angles at A_1 and A_2 respectively, while R is on the exterior bisector of A_3; the equation is satisfied by the perpendiculars, but the triangles are not in perspective. The same flaw vitiates the current proofs of several other standard theorems of this type.

For if XP_2 and XP_3 are homologous lines of II and III, meeting at X on the circle of similitude, and cutting that circle again at T_2 and T_3 respectively, then each of the angles $\measuredangle C_1XP_2$, $\measuredangle C_1XP_3$ is constant and therefore arcs C_1T_2 and C_1T_3 are constant, and T_2, T_3 are fixed points.

THEOREM 11. *The invariable points are homologous; they form a triangle inversely similar to that formed by three homologous lines.*

For (*a*) triangles $C_1T_2T_3$ and $C_2P_2P_3$ are similar, (*b*) $\measuredangle T_1T_3T_2 = \measuredangle T_1XT_2$, etc. Let C_2T_2 meet C_3T_3 at M.

THEOREM 12. *The point C_1', homologous in I to the double point C_1 of II and III, lies on the circle C_2C_3M.*

For

$$\measuredangle C_2C_1'T_1 = \measuredangle C_2C_1T_3 = \measuredangle C_2T_2T_3$$

$$\measuredangle T_1C_1'C_3 = \measuredangle T_2C_1C_3 = \measuredangle T_2T_3C_3$$

whence

$$\measuredangle C_2C_1'C_3 = \measuredangle C_2T_2,\ C_3T_3.$$

THEOREM 13. *Both C_1' and M lie on C_1T_1.*

For

$$\measuredangle C_3T_1C_1' = \measuredangle C_3T_3C_1 = \measuredangle C_3T_1C_1$$

and

$$\measuredangle C_3C_1'M = \measuredangle C_3C_2M = \measuredangle C_3C_2T_2 = \measuredangle C_3T_1T_2 = \measuredangle C_3C_1'T_1.$$

Our results may be summed up as follows:

The invariable triangle and the triangle of similitude are inscribed in the same circle and are in perspective at a point M. The lines C_1T_1, C_2T_2, C_3T_3, which are concurrent at M, meet again the respective circles C_2C_3M, C_3C_1M, C_1C_2M at C_1', C_2', C_3' respectively.

Concerning the properties of triads of homologous points, we merely state the main theorems; proofs by means of directed angles are not difficult.

THEOREM 14. *If P_1, P_2, P_3 are homologous, circles $C_2C_3P_1$, $C_3C_1P_2$, $C_1C_2P_3$ are concurrent at a point X, for which*

$$\measuredangle C_2XC_3 = \measuredangle C_2MC_3 + \measuredangle P_3P_1P_2.$$

COROLLARY. If homologous points P_1, P_2, P_3 move so that $\measuredangle P_3P_1P_2$ remains equal to a given angle α, then each moves on a circle; in particular, P_1 lies on that circle through C_2 and C_3 for which

$$\measuredangle C_2P_1C_3 = \measuredangle C_2MC_3 + \alpha.$$

THEOREM 15. *The loci of collinear homologous points are respectively the circles C_2C_3M, C_3C_1M, C_1C_2M; and the line through the three points passes also through M.*

THEOREM 16. *There is a single triad of homologous points which are the vertices of a triangle similar in a given sense to a given triangle.*

The construction is effected as follows: let $V_1V_2V_3$ be the given triangle. Let X be so located that

$$\measuredangle C_2XC_3 = \measuredangle C_2MC_3 + \measuredangle V_3V_1V_2$$

with similar expressions for $\measuredangle C_3XC_1$ and $\measuredangle C_1XC_2$. There is just one such point X. Now regard circle C_2XC_3 as a part of Fig. 1, and find the homologous circle of II, then P_2, one of the required triad of points, lies on this circle. But P_2 lies also on circle C_3XC_1, and therefore it is uniquely determined.

THEOREM 17. *Similarly, there is a single triad of collinear homologous points, whose distances are in the same ratios as the distances among three given collinear points.*[1]

It is well known that in the special case when three similar figures are constructed on the sides of a triangle $A_1A_2A_3$, in such a way that A_2A_3, A_3A_1, A_1A_2 are homologous segments, then

(a) the triangle of similitude is the second Brocard triangle,
(b) the invariable triangle is the first Brocard triangle,
(c) the circle of similitude is therefore the Brocard circle,
(d) the point M is the median point,
(e) the circles MC_2C_3, etc., are the Neuberg circles.

NOTE ON A CERTAIN CLASS OF DETERMINANTS.

By W. H. METZLER, Syracuse University.

In this journal for May, 1915, there appeared the translation of a paper by Professor Pascal which considered a class of determinants which have the property of being expressible as the sum of two squares, and it is the main object of this note to point out another proof of this fact.

The determinants in question are of the form:

$$D \equiv \begin{vmatrix} a_{11} \cdots a_{1n} & -b_{11} \cdots -b_{1n} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{n1} \cdots a_{nn} & -b_{n1} \cdots -b_{nn} \\ b_{11} \cdots b_{1n} & a_{11} \cdots a_{1n} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ b_{n1} \cdots b_{nn} & a_{n1} \cdots a_{nn} \end{vmatrix},$$

which is the determinant of the set of linear homogeneous complex equations for the $2n$ quantities $p_1, \cdots, p_n, q_1, \cdots, q_n$:

$$\begin{aligned} &(a_{11} + b_{11}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{1n} + b_{1n}\cdot i)(p_n + q_n\cdot i) = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &(a_{n1} + b_{n1}\cdot i)(p_1 + q_1\cdot i) + \cdots + (a_{nn} + b_{nn}\cdot i)(p_n + q_n\cdot i) = 0. \end{aligned} \tag{1}$$

[1] Theorems 16 and 17 are doubtless well known, but the author has not chanced to see them explicitly stated. But see McClelland.

The determinant D is obviously unaltered in value[1] by rearranging the last n rows and the last n columns so that they will be in the reverse order, and then multiplying the elements of the first n rows and the last n columns by i. In this form it is seen to be centro-symmetric with all the b's positive and all the a's multiplied by i. It follows, therefore, that it is the product of two factors D' and D'', say, and it has been shown[2] that one of these factors is the sum of two sets of terms and the other is the difference of the same two sets.

Thus

$$D' \equiv \begin{vmatrix} b_{1n} + a_{1n} \cdot i & \cdots & b_{11} + a_{11} \cdot i \\ \cdot & \cdots & \cdot \\ b_{nn} + a_{nn} \cdot i & \cdots & b_{n1} + a_{n1} \cdot i \end{vmatrix}, \quad \text{or} \quad | b_{rs} + a_{rs} \cdot i | ;$$

and

$$D'' \equiv \begin{vmatrix} b_{1n} - a_{1n} \cdot i & \cdots & b_{11} - a_{11} \cdot i \\ \cdot & \cdots & \cdot \\ b_{nn} - a_{nn} \cdot i & \cdots & b_{n1} - a_{n1} \cdot i \end{vmatrix}, \quad \text{or} \quad | b_{rs} - a_{rs} \cdot i | .$$

As D' and D'' are determinants with binomial elements, each may be expanded into a sum of determinants with monomial elements giving

$$D = (X + i \cdot Y)(X - i \cdot Y)$$
$$= X^2 + Y^2.$$

Representing the matrix of D symbolically by

$$\left\| \begin{matrix} (A) & (B) \\ (-B) & (A) \end{matrix} \right\|,$$

Professor Pascal remarks that "it still remains to study determinants whose matrices are of the type

$$\left\| \begin{matrix} (A) & (B) & (C) & \cdots \\ (-B) & (A) & (D) & \cdots \\ (-C) & (-D) & (A) & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{matrix} \right\| ."$$

If our set of equations were of the form:

$$\begin{aligned} &(a_{11} + b_{11} \cdot i)(p_1 + q_1 \cdot i) + \cdots + (a_{1n} + b_{1n} \cdot i)(p_n + q_n \cdot i) = 0, \\ &-(a_{12} + b_{12} \cdot i)(p_1 + q_1 \cdot i) + \cdots + (a_{2n} + b_{2n} \cdot i)(p_n + q_n \cdot i) = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &-(a_{1n} + b_{1n} \cdot i)(p_1 + q_1 \cdot i) - \cdots + (a_{11} + b_{11} \cdot i)(p_n + q_n \cdot i) = 0, \end{aligned} \tag{2}$$

[1] The sign will be changed if n is odd.

[2] Metzler, On Centro-Symmetric and Skew-Centro-Symmetric Determinants, *Messenger of Mathematics*, New Series, No. 515, March, 1914.

where the diagonal coefficients are all $a_{11} + b_{11}i$, the determinant D would be

$$D = \begin{vmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} & \cdots & a_{1n} & -b_{1n} \\ b_{11} & a_{11} & b_{12} & a_{12} & \cdots & b_{1n} & a_{1n} \\ -a_{12} & b_{12} & a_{11} & -b_{11} & \cdots & a_{2n} & -b_{2n} \\ -b_{12} & -a_{12} & b_{11} & a_{11} & \cdots & b_{2n} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1n} & b_{1n} & -a_{2n} & b_{2n} & \cdots & a_{11} & -b_{11} \\ -b_{1n} & -a_{1n} & -b_{2n} & -a_{2n} & \cdots & b_{11} & a_{11} \end{vmatrix};$$

or expressed symbolically

$$D = \begin{Vmatrix} (A) & (B) & (C) & \cdots \\ (-B) & (A) & (D) & \cdots \\ (-C) & (-D) & (A) & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix},$$

the form in question.

By a rearrangement of rows and columns it is readily seen that D may be represented symbolically by

$$\begin{Vmatrix} (A') & (B') \\ (-B') & (A') \end{Vmatrix}.$$

This is what should be expected since the D for equations (2) is a particular case of the D for equations (1).

If the $A, B, C, \cdots$ here are to be of higher order than the second it will be necessary to take equations from algebras of higher order, such as quaternions for the fourth order.

THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America held its fall meeting at St. John's College, Annapolis, Md., Saturday, December 15, 1917. Among the thirty-two persons in attendance were the following members of the Association:

O. S. Adams, U. S. Coast and Geodetic Survey, Washington, D. C.
J. A. Bullard, U. S. Naval Academy, Annapolis, Md.
Paul Capron, U. S. Naval Academy, Annapolis, Md.
G. R. Clements, U. S. Naval Academy, Annapolis, Md.
A. B. Coble, Johns Hopkins University, Baltimore, Md.
A. Cohen, Johns Hopkins University, Baltimore, Md.
J. B. Eppes, U. S. Naval Academy, Annapolis, Md.
J. N. Galloway, U. S. Naval Academy, Annapolis, Md.
H. C. Gossard, U. S. Naval Academy, Annapolis, Md.
Angelo Hall, U. S. Naval Academy, Annapolis, Md.

W. M. Hamilton, Nautical Almanac Office, Washington, D. C.
W. E. Heal, Washington, D. C.
W. W. Johnson, U. S. Naval Academy, Annapolis, Md.
A. E. Landry, Catholic University, Washington, D. C.
Florence P. Lewis, Goucher College, Baltimore, Md.
H. M. Robert, U. S. Naval Academy, Annapolis, Md.
R. E. Root, U. S. Naval Academy, Annapolis, Md.
W. F. Shenton, U. S. Naval Academy, Annapolis, Md.
H. I. Thomsen, Baltimore, Md.
H. R. Tolley, Department of Agriculture, Washington, D. C.
C. E. Van Orstrand, U. S. Geological Survey, Washington, D. C.

The president of the section, Professor Abraham Cohen, occupied the chair at both morning and afternoon sessions, and the following papers were presented:

1. "Report on the 1917 summer meeting of the Association at Cleveland." DR. G. R. CLEMENTS, U. S. Naval Academy.
2. "A method for finding a particular integral for certain linear differential equations." PROFESSOR A. B. COBLE, Johns Hopkins University.
3. "Report on the construction of certain mathematical tables." MR. C. E. VAN ORSTRAND, U. S. Geological Survey.
4. "Geometrical stereograms." DR. W. F. SHENTON, U. S. Naval Academy.
5. "Lambert's conformal conic projection." MR. O. S. ADAMS, U. S. Coast and Geodetic Survey.
6. "An application of Grassmann's Ausdehnungslehre to the theory of invariants." MR. W. E. HEAL, Washington, D. C.
7. "Pseudo-regular polyhedra." PROFESSOR W. W. JOHNSON, U. S. Naval Academy.

A luncheon at the Hotel Maryland, arranged for those attending these sessions, was participated in by twenty-two persons.

RALPH E. ROOT,
Secretary.

BOOK NOTICES.

SEND COMMUNICATIONS ABOUT BOOKS TO W. H. BUSSEY, University of Minnesota.

In 1915 John Wiley and Sons published an *Analytic Geometry* by Professor H. B. Phillips of the Massachusetts Institute of Technology and in 1916 a *Differential Calculus* by the same author. These have been reviewed in this MONTHLY (see Volume 23, page 17 and Volume 24, page 78). The course in mathematics has now been extended by the publishing of an *Integral Calculus* which may be purchased separately or bound with the *Differential Calculus*. It contains answers to problems, a short table of integrals and a table of natural logarithms.

Every mathematician knows that many a problem is hard or easy according

as one system of coördinates is used or another. For example the Desargues theorem about perspective triangles is hard if rectangular Cartesian coördinates are used and easy by means of trilinear coördinates. Students of mathematical physics will be interested in *An Introduction to the Use of Generalized Coördinates in Mechanics and Physics*, by Professor W. E. Byerly of Harvard University, published by Ginn and Company in 1916.

The geometry of hyperspace is not usually thought of as a part of applied mathematics, but Mr. Claude Bragdon, an architect, has made use of his knowledge of the geometry of four dimensions in the making of new designs for architectural purposes in a book entitled *Projective Ornament*, published in 1915 by The Manas Press, Rochester, N. Y. He has extended this work in an article entitled "Art and Geometry" in the *Architectural Review*, New Series, Volume 4, pages 29–32, 58–61. Mr. Bragdon is also the author of *A Primer of Higher Space*, published by The Manas Press in 1913, and *Four-dimensional Vistas*, published by Alfred A. Knopf, New York, 1916. His interest in higher space is largely mystical, but he is well acquainted with the standard mathematical books on the subject and he has made some practical use of it.

Those who were at the annual meeting of the Association in Chicago on December 27, 28 will be interested in knowing that Professor Roever of Washington University, who gave an address on Descriptive Geometry, is going to write for the MONTHLY a review of a new book on descriptive geometry by Ervin Kenison and Harry Cyrus Bradley, of the Massachusetts Institute of Technology, which has just been published by The Macmillan Company in the series of texts on topics in engineering edited by Earle Raymond Hedrick. Another new book in this same series, in which some of the readers of the MONTHLY may be interested, is *Electric and Magnetic Measurements* by Professor Charles Marquis Smith of the physics department of Purdue University.

Perhaps it is not well known that a supplement to the well-known book on *The Mathematical Theory of Investment* by Professor C. E. Skinner was prepared by Dr. G. R. Clements and published under the title *Problems in the Mathematical Theory of Investment* by Ginn and Company in 1916. The little book sells for 32 cents.

In 1914 the McGraw-Hill Book Company published *Mathematics for Agricultural Students* by Henry C. Wolff. Another book with a very similar title has recently come from the press, namely, *Mathematics for Collegiate Students of Agriculture and General Science*, by A. M. Kenyon and W. V. Lovitt of Purdue University. It is one of the series of mathematical texts edited by E. R. Hedrick and published by The Macmillan Company. It sells for $2.00.

In 1910 The Century Company published a text-book on *Advanced Algebra and Trigonometry* by W. C. Brenke, which, according to its preface, treated both subjects in a correlated manner. But according to a review in the *Bulletin of the American Mathematical Society*, Volume 16, page 540, the degree of correlation was not very great. So the recent publication of two separate volumes by W. C. Brenke entitled *Advanced Algebra* and the *Elements of Trigonometry* respectively

is not really a case of uncorrelating the correlated. The *Advanced Algebra* contains in revised form and with some additions the material of the chapters on algebra in the original Brenke book, and a similar statement is true of the *Trigonometry*. The books may be had separately or bound together in one volume.

Correlation has been talked about for many years and some books on unified mathematics have been published. But most of the teaching of mathematics in this country is still done by means of the more or less old-fashioned text-books on algebra, trigonometry, etc., as separate subjects. The latest attempt at a correlation of college algebra, trigonometry and analytic geometry is the new *Elementary Mathematical Analysis* by J. W. Young and F. M. Morgan of Dartmouth College. Later it will be reviewed at length in the MONTHLY.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2680. Proposed by C. C. YEN, Tangshan, North China.

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. [From Joseph Edward's *Elementary Treatise on Differential Calculus*.]

2681. Proposed by PHILIP FRANKLIN, College of the City of New York.

Given n letters of one kind and $n-1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?

2682. Proposed by E. L. REES, University of Kentucky.

Given the diagonal and the angle it makes with a bisector of one of the angles of a parallelogram. Construct the parallelogram so that the rectangle having sides equal to those of the parallelogram may have a given area.

2683. Proposed by J. R. HITT, Student, Mississippi College.

The height of a frustum of a cone is h, the radii of the lower and upper circular bases are a and b, respectively. Deduce a formula for finding the center of gravity of the frustum.

2684. Proposed by B. J. BROWN, Kansas City, Mo.

Find the locus of the center of a conic passing through four fixed points.

2685. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

A particle is describing an ellipse of eccentricity $\sqrt{2/3}$ as a central orbit about a focus when the attracting force suddenly becomes repulsive without changing its magnitude and the particle begins to describe an equilateral hyperbola; find where the change occurred and the angle that the major axis of the new orbit makes with that of the old orbit.

2686. Proposed by EDWIN R. SMITH, State College, Pa.

Given the difference equation

$$u_{x+1} - u_x = \log\left(1 - \frac{x}{sp}\right) - \log\left(1 + \frac{x+1}{sq}\right).$$

where s is a positive integer, p and q are proper fractions such that $p + q = 1$, and x is small when compared with sp and sq. Determine T_x, if $u_x = \log T_x$.

2687. Proposed by N. P. PANDYA, Sojitra, India.

An ellipse intersects a parabola in A and B, and the tangents at A and B to the parabola meet at T. The center C of the ellipse lies within the space enclosed by the parabola and the tangents. Draw a third tangent to the parabola such that C may be the centroid of the triangle formed by the three tangents.

2688. Proposed by FRANK IRWIN, University of California.

With four quantities, a_1, a_2, a_3, a_4, we may, without changing their order, form the following complex fractions:

$$\frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}} \qquad \frac{\frac{a_1}{a_2}}{\frac{a_3}{a_4}}$$

But these have not all different values; the first and fourth are equal.

Determine how many different rational functions of the quantities $a_1, a_2, \cdots a_n$ may be obtained in this way, and which can be represented in more than one way as a complex fraction of the above kind, and which in only one way.

Restatement of 434 (Calculus). Proposed by E. W. CHITTENDEN, Champaign, Ill.

Evaluate $\int_0^1 f(x)dx$ where

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\operatorname{sgn}(x - x_n)}{2^n}.$$

The function $\operatorname{sgn} x = -1, 0, +1$ according as x is negative, zero or positive. The numbers x_n form the series

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \cdots$$

for which the general formula is $x_n = (2h + 1)/2^k$, where k is the greatest integer such that $2^{k-1} \leqq n$ and $h = n - 2^{k-1}$.

Note. This problem is reprinted because of a misprint in the original statement and also because of a lack of definiteness in the definition of the series of values x_n. An inquiry has also been received in regard to the origin of the notation "sgn x." This notation seems to be due to Kronecker; cf. Werke, Vol. II, p. 500.

SOLUTIONS OF PROBLEMS.

487 (Algebra). Proposed by WILLIAM HOOVER, Columbus, Ohio.

Show in two ways that 0.5623 is not a root of $(1 - m)e^m = 2e^{-1}$, e being the Napierian base. Find the value of m, and test the result in two ways.

SOLUTION BY E. E. WHITFORD, New York City.

Taking logarithms of both sides, we have

$$\log(1 - m) + (m + 1)\log e = \log 2, \quad \text{or} \quad \log(1 - m) = \log 2 - 1 - m.$$

Substituting .5623 for m,

$$\log .4377 = .6931 - 1 - .5623 = 9.1308 - 10.$$

But this is false since $\log .4377 = 9.1733 - 10$. Or, from $(1 - m)e^{m+1} = 2$, on taking $m = .5623$ we have $e^{1.5623} = 4.57$. But this is false, since $4.57 = e^{1.5195}$. One value of m is .5936 approximately; this checks by both the above methods. We may also have $m = -1$.

Also solved by C. E. FLANAGAN and EDWARD H. WORTHINGTON.

488 (Algebra). Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0,$$

where

$$a_k = \frac{\sin(k\theta + \alpha)}{\sin k\theta},$$

and θ and α have any values that do not make a denominator zero.

SOLUTION BY A. M. HARDING, University of Arkansas.

It is easily shown that the following equations hold for all values of θ:

$$\cot\theta - 2\cot 2\theta + \cot 3\theta = \frac{2\cos 2\theta}{\sin 4\theta + \sin 2\theta},$$

$$\cot\theta\cot 3\theta - \cot^2 2\theta = \frac{2\cos 2\theta\cot 4\theta}{\sin 4\theta + \sin 2\theta};$$

that is

$$\cot\theta\cot 3\theta - \cot^2 2\theta = \cot 4\theta(\cot\theta - 2\cot 2\theta + \cot 3\theta).$$

Now a_k may be written in the form

$$a_k = \cos\alpha + \sin\alpha\cot k\theta.$$

Thence we have

$$a_1 - 2a_2 + a_3 = \sin\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta),$$

and

$$\begin{aligned} a_1a_3 - a_2^2 &= \sin\alpha\cos\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta) + \sin^2\alpha(\cot\theta\cot 3\theta - \cot^2 2\theta) \\ &= \sin\alpha(\cot\theta - 2\cot 2\theta + \cot 3\theta)(\cos\alpha + \sin\alpha\cot 4\theta) \\ &= (a_1 - 2a_2 + a_3)a_4. \end{aligned}$$

Hence

$$\frac{a_1a_3 - a_2^2}{a_4} + 2a_2 - a_1 - a_3 = 0;$$

that is,

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0.$$

Also solved by PAUL CAPRON, OSCAR ADAMS, and EDWARD H. WORTHINGTON.

489 (Algebra). Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a\begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b\begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab\begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

SOLUTION BY J. L. RILEY, Stephenville, Texas.

Evaluating the first determinant, we have $[R(R - 2ay^2 - 2bu^2)]^3$, where

$$R = x^2 + ay^2 + bu^2 + abv^2.$$

The second determinant, when expanded, gives $a[-2R(xy + buv)]^3$; the third gives $b[2R(avy - ux)]^3$; the fourth gives 0; and the fifth determinant gives $[-R^2]^3$. But

$$[R(R - 2ay^2 - 2bu^2)]^3 + a[-2R(xy + buv)]^3 + b[2R(avy - ux)]^3 \neq [-R^2]^3$$

unless $(ay^2)(bu^2) = 0$. Hence the relation stated in the problem does not always hold.

520 (Geometry). Proposed by ALBERT A. BENNETT, University of Texas.

On a given tangent to a circle determine a point such that, if a secant be drawn joining this point to the extremity of the diameter which is perpendicular to the given tangent, the segment of this secant exterior to the circle will be equal in length to a given segment.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let us denote the radius of the given circle by r and the length of the given segment by $2d$. Let AB be the diameter perpendicular to the tangent at the point of tangency A. Take a length $AC = d$ along the tangent from A. Join BC. Take D on CB so that $CD = d$. With center at B draw arc DE cutting the circle at E. Produce BE to cut the tangent at P. Then P is the required point.

Proof:

$$BE = BD = \sqrt{d^2 + 4r^2} - d.$$

Since AE is perpendicular to BP it follows that

$$AB^2 = BE \times BP = BE(BE + EP).$$

Hence,

$$4r^2 = (\sqrt{d^2 + 4r^2} - d)(\sqrt{d^2 + 4r^2} - d + EP).$$

From this equation we find

$$EP = 2d.$$

Note: If a point Q be taken on the tangent such that $AQ = AP$, this point Q will also satisfy the conditions of the problem.

Also solved by MAY PHALOR, H. T. AUDE, HERBERT N. CARLETON, OSCAR S. ADAMS, H. C. FEEMSTER, and PAUL CAPRON.

521 (Geometry). Proposed by R. M. MATHEWS, Riverside, Cal.

A variable circle, with center on the line l and passing through a fixed point P, cuts a fixed circle in A and B. Prove that the common chord AB and the perpendicular to l through P intersect in a fixed point.

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

It is convenient to employ rectangular coördinates. Let l be taken as the axis of x and the point P on the y-axis; then the perpendicular to l through P is the axis of y. Under these conditions, it follows from elementary analytic geometry that the equation of the variable circle is

$$(x - \alpha)^2 + y^2 = r^2,$$

where r is the radius of the variable circle. The equation of the fixed circle may be taken as

$$(x - a)^2 + (y - b)^2 = c^2.$$

Subtracting the first equation from the second, we get

$$2(\alpha - a)x - 2by = c^2 - a^2 + \alpha^2 - r^2 - b^2,$$

which is the equation of a straight line through A and B. To find the point in which this line cuts the axis of y, set x equal to zero and solve the resulting equation for y. Since $r^2 - \alpha^2$ is independent of the position of the variable circle, it follows that the intersection of the chord AB and the perpendicular to l through P is fixed.

It is interesting to note that the above proof still holds when the intersections of the two circles are imaginary points.

Also solved by H. C. FEEMSTER, OSCAR S. ADAMS, HAROLD R. SCHAUFLER, A. M. HARDING, PAUL CAPRON, MARGARET F. WILLCOX, and ROGER A. JOHNSON.

522 (Geometry). Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the middle points of the opposite edges.

SOLUTION BY R. M. MATHEWS, Riverside, California.

Let $ABCD$ be the tetrahedron with X, the mid-point of BD, opposite to Y, the mid-point of AC. In the triangle BYD, we have

$$2XY^2 = BY^2 + DY^2 - 2BX^2,$$

from the theorem: The sum of the squares on two sides of a triangle is equal to twice the square on half the third side plus twice the square on the median to that side.

By the same theorem, we find $2BY^2$ in the triangle ABC and $2DY^2$ in the triangle ADC. Then,

$$4XY^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2.$$

Similarly with U, the middle point of AB, and V, of DC:

$$4UV^2 = AC^2 + CB^2 + BD^2 + DA^2 - AB^2 - DC^2;$$

and with T, the middle point of AD, and W, of BC:

$$4TW^2 = AC^2 + CD^2 + DB^2 + BA^2 - AD^2 - BC^2.$$

By addition, we have

$$4(XY^2 + UV^2 + TW^2) = AB^2 + BC^2 + CD^2 + AD^2 + AC^2 + BD^2.$$

Remark: Considering any one of the three equations above we have proved the theorem: Four times the square on the median joining two opposite edges of a tetrahedron is equal to the sum of the squares on the other edges minus the sum of the squares on the edges which it joins.

Also solved by HORACE OLSON, OSCAR S. ADAMS, J. L. RILEY, and H. C. GOSSARD.

345 (Mechanics). Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Two particles A and B are together in a smooth circular tube. A attracts B with a force whose acceleration is ω^2 and moves along the tube with uniform angular velocity 2ω, B being initially at rest; prove that the angle φ subtended by AB at the center after a time t is given by the equation

$$\log \tan \frac{\pi + \varphi}{4} = \omega t.$$

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let B' be the initial positions of A and B, and put $BB' = s$, and $AB = x$.

Resolving forces tangentially,

$$\frac{d^2s}{dt^2} = \omega^2 x \cos ABC, \tag{1}$$

BC being the tangent through B.

Now if O be the center of the circle,

$$\angle AOB = \frac{2r\omega t - s}{r}, \qquad \angle ABC = \frac{2r\omega t - s}{2r},$$

$$x \cos ABC = 2r \sin \tfrac{1}{2}AOB \cos ABC = 2r \sin \frac{2r\omega t - s}{2r} \cos \frac{2r\omega t - s}{2r} = r \sin \frac{2r\omega t - s}{r}$$

$$= r \sin (2\omega t - \theta), \ \theta \text{ being the angle } BOB' \text{ and } s = r\theta.$$

Substituting these in (1),

$$\frac{d^2(2\omega t - \theta)}{dt^2} = -\omega^2 \sin(2\omega t - \theta), \tag{2}$$

or, since $2\omega t - \theta = \varphi$,

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \sin\varphi. \tag{3}$$

Multiplying by $2(d\varphi/dt)$ and integrating,

$$\frac{d\varphi^2}{dt^2} = 2\omega^2 \cos\varphi + C. \tag{4}$$

When $\varphi = 0$, $d\varphi/dt = 0$; hence, $C = 2\omega^2$, and (4) is

$$\frac{d\varphi^2}{dt^2} = 2\omega^2(1 + \cos\varphi) = 4\omega^2 \cos^2 \tfrac{1}{2}\varphi. \tag{5}$$

This gives

$$\omega dt = \frac{d\tfrac{1}{2}\varphi}{\cos\tfrac{1}{2}\varphi}; \tag{6}$$

integrating and using the fact that $\varphi = 0$ when $t = 0$, we have

$$\log\tan\frac{\pi + \varphi}{4} = \omega t. \tag{7}$$

Also solved by G. PAASWELL and J. B. REYNOLDS.

346 (Mechanics). Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

SOLUTION BY THE PROPOSER.

Let m = the mass of a unit of length of the string, $2a$ = the whole length of the string, and so a, $a/2$, the parts at the instant of cutting, $3a/2$ the length in motion after any time t from the beginning of motion, x = the longer part at the same instant, and T = the corresponding tension in the string; then the equations of motion are, noticing that the momenta of the moving masses are each variable,

$$\frac{d}{dt}\left(mx\frac{dx}{dt}\right) = m\left(\frac{dx^2}{dt^2} + x\frac{d^2x}{dt^2}\right) = mgx - T, \tag{1}$$

$$\frac{d}{dt}\left\{m(\tfrac{3}{2}a - x)\frac{d}{dt}(\tfrac{3}{2}a - x)\right\} = \frac{d}{dt}\left\{-m\tfrac{3}{2}a\frac{dx}{dt} + mx\frac{dx}{dt}\right\} = mg(\tfrac{3}{2}a - x) - T. \tag{2}$$

Subtracting (2) from (1), we have

$$\frac{d^2x}{dt^2} = \frac{2g}{3a}\left(2x - \frac{3a}{2}\right). \tag{3}$$

Multiplying by $2(dx/dt)$ and integrating, noticing that when $x = a$, $(dx/dt) = 0$, we obtain

$$\frac{dx^2}{dt^2} = \frac{g}{3a}\left\{\left(2x - \frac{3a}{2}\right)^2 - \frac{a^2}{4}\right\}. \tag{4}$$

Substituting the values of dx^2/dt^2 and d^2x/dt^2 in (1), and putting $T = T_0$ when $x = a$, we find $T_0 = \frac{2}{3}mga$.

If P = the required initial pressure, $P = 2T_0 = \frac{4}{3}mga = 2mga - \frac{1}{3}\cdot 2mga$, so that the pressure before cutting is diminished by one third.

Also solved by HORACE OLSON.

264 (Number Theory). Proposed by C. F. GUMMER, Kingston, Canada.

Find a general formula for three squares in arithmetical progression. Is it possible for the common difference to be a perfect square?

Solution by Artemas Martin, LL.D., Washington, D. C.

Let x^2, y^2, z^2 be three square numbers in arithmetical progression; then we must have

$$y^2 - x^2 = z^2 - y^2, \quad \text{or} \quad x^2 + z^2 = 2y^2. \tag{1}$$

Assume $z = v + w$, $x = v - w$, and (1) becomes after dividing by 2,

$$v^2 + w^2 = y^2. \tag{2}$$

Take now $v = p^2 - q^2$, $w = 2pq$, and (2) is satisfied. Retracing, we find

$$z = p^2 - q^2 + 2pq, \quad x = p^2 - q^2 - 2pq, \quad y = p^2 + q^2.$$

Hence, the required squares are

$$x^2 = (p^2 - q^2 - 2pq)^2, \quad y^2 = (p^2 + q^2)^2, \quad z^2 = (p^2 - q^2 + 2pq)^2.$$

Taking $p = 2$, $q = 1$, the numbers are 1, 25, 49; taking $p = 3$, $q = 2$, the numbers are 49, 169, 289; taking $p = 4$, $q = 1$, the numbers are 49, 289, 529; taking $p = 4$, $q = 3$, the numbers are 289, 625, 961; and so on, indefinitely.

The common difference of three square numbers in arithmetical progression can not be a square number. See Barlow's "Theory of Numbers," p. 257. An equivalent theorem is also given in Carmichael's *Diophantine Analysis*, p. 14.

There can not be *four* square numbers in arithmetical progression. Barlow, same page. Therefore there can not be *five*, nor any greater number than three, of squares in arithmetical progression.

Also solved by J. L. Riley, V. M. Spunar, and the Proposer.

265 (Number Theory). Proposed by J. W. Nicholson, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)^2$ is a perfect square, y and z being any two roots of the equation.

Solution by N. P. Pandya, Sojitra, India.

Since y and z are roots of the given equation, $x^2 - x(y + z) + yz$ is a factor of the left-hand side of the equation.

Since the term in x^3 is wanting, the remaining roots are given by

$$x^2 + x(y + z) + \frac{c}{yz} = 0. \tag{1}$$

The product of (1) with $x^2 - x(y + z) + yz = 0$ gives

$$a = -\frac{c}{yz} - yz + (y + z)^2.$$

Hence,

$$4(a + yz) = -\frac{4c}{yz} + 4(y + z)^2,$$

or

$$4(a + yz) - 3(y + z)^2 = (y + z)^2 - \frac{4c}{yz} = \text{a square},$$

since the roots of (1) are rational and its discriminant is therefore a square.

267 (Number Theory). Proposed by C. C. Yen, Tangshan, North China.

A number theory function $\phi(n)$ is defined for every positive integer n, and for every such number n it satisfies the relation $\phi(d_1) + \phi(d_2) + \phi(d_3) + \cdots + \phi(d_r) = n$, where $d_1, d_2, \cdots, d_r$ are the divisors of n. From this property alone show that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p \quad p \quad p_2 \cdots p_k$ are the different prime factors of n.

SOLUTION BY OLIVE C. HAZLETT, Bryn Mawr College.

The theorem is clearly true for all primes. Accordingly, assume the theorem is true for all divisors of $n = \prod_{i=1}^{k} p_i^{e_i}$ which are less than n. Now for n the defining equation becomes

$$\phi(n) + A_n + p_1^{e_1}\left(1 - \frac{1}{p_1}\right) B_n = n,$$

where A_n is the sum of the ϕ-functions formed for the divisors of $p_1^{e_1-1} p_2^{e_2} \cdots p_k^{e_k}$ and B_n is a similar sum formed for all distinct factors of any of the numbers $p_2^{e_2-1} p_3^{e_3} \cdots p_k^{e_k}$, $p_2^{e_2} p_3^{e_3-1} p_4^{e_4} \cdots p_k^{e_k}$, $\cdots$. It is easy to find an expression for A_n, but it is sufficient for our purposes to note that A_n is a polynomial in $p_2, \cdots, p_k$ of degree at most $\sum_{i=2}^{k} e_i - 1$. Therefore $p_1^{e_1}\left(1 - \frac{1}{p_1}\right)$ is a factor of $\phi(n)$. Since this proof is perfectly general, it holds for every expression of the form $p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ $(i = 1, \cdots, p)$, and thus $\prod_{i=1}^{k} p_i^{e_i}\left(1 - \frac{1}{p_i}\right)$ is a factor of $\phi(n)$. Comparing the coefficients of $\prod_{i=1}^{k} p_i^{e_i}$ our formula is proved.

Also solved by H. C. FEEMSTER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO FINDING DERIVATIVES OF TRIGONOMETRICAL FUNCTIONS.

By T. H. HILDEBRANDT, University of Michigan.

In most textbooks on the elementary calculus the derivatives of the trigonometric functions are based on the derivative of the sine function, which, in turn, is derived from the definition of derivative. The proofs dealing with the value of this derivative seem to have something indirect about them. All goes well until the point is reached where the expression

$$\lim_{\Delta x \doteq 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

is to be evaluated, and then one of two methods is used. Either $\sin(x + \Delta x)$ is expanded by the formula for the sine of the sum of two angles and the formula for $1 - \cos x$ in terms of half angles is used, or the formula for the difference of two sines is used. Both of these latter formulæ have long since escaped the memory of the average sophomore student—if they ever had lodging there—and he practically accepts this part of the derivation on faith.

While it must be admitted that the most natural beginning for a chapter on the derivatives of trigonometrical functions is a paragraph devoted to finding the derivative of the sine, this advantage is more than counterbalanced by the simplicity with which it is possible to obtain the derivative of the tangent function directly from the definition of derivative—a fact which seems almost to have

escaped the attention of writers of textbooks on calculus. For this purpose it is possible to proceed in either of two ways, both of which are elegant and altogether natural and direct. If we take $\tan x = \sin x/\cos x$, then

$$\tan(x+\Delta x) - \tan x = \frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\Delta x)\cos x - \cos(x+\Delta x)\sin x}{\cos(x+\Delta x)\cos x}$$

$$= \frac{\sin \Delta x}{\cos(x+\Delta x)\cos x}.$$

If we divide now by Δx and take the limit as Δx approaches zero, then $\lim_{\alpha \doteq 0} (\sin \alpha)/\alpha = 1$ is applicable for evaluating the derivative. Or, proceeding directly, we have

$$\tan(x+\Delta x) - \tan x = \frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x$$

$$= \frac{(1+\tan^2 x)\tan \Delta x}{1 - \tan x \tan \Delta x}.$$

Then by using the fact that $\lim_{\alpha \doteq 0} (\tan \alpha)/\alpha = 1$, we again get at once the value of the derivative of the tangent. Either of these methods yields this derivative without any troublesome trigonometrical transformations.

On the basis of the derivative of the tangent, the remaining derivatives are easily obtained in the order $\sec x$, $\cos x$, and $\sin x$: the secant by taking derivatives in the relation

$$\tan^2 x + 1 = \sec^2 x,$$

the cosine through the fact that it is the reciprocal of the secant, and the sine from one of the relations

$$\sin x = \cos(\pi/2 - x), \qquad \sin x = \cos x \tan x.$$

We get, then, in this way the derivatives which are used most, and we employ only trigonometrical relations which are familiar to the average sophomore student.

II. Relating to the Proof that a Rigid Body Moving about a Fixed Point Is at Each Instant Rotating about an Axis through that Point.

By E. L. Rees, University of Kentucky.

The following proof of this theorem differs from other proofs by vector methods in that it is based upon an interesting geometric interpretation of the equations involved.

Let O be the fixed point, and P and Q any two points of the body distinct from O, and let $OP = p$ and $OQ = q$. Using the Gibbs notation we have as the assumptions of rigidity $p^2 =$ constant, $q^2 =$ const., $p \cdot q =$ const., or differentiating with respect to t,

$$p \cdot \dot{p} = 0, \qquad q \cdot \dot{q} = 0, \qquad p \cdot \dot{q} + \dot{p} \cdot q = 0, \tag{1}$$

where $\dot{p}$ and $\dot{q}$ are the velocities of P and Q respectively.

Consider the equations $\dot{p} = w \times p$ and $\dot{q} = w \times q$. Regarding w as the running coordinate, each of these equations represents a line parallel to the position vector and perpendicular to the velocity of the point. If $p \times q = 0$, *i. e.*, if O, P, and Q are collinear, the lines coincide. For, substituting $p = kq$ in $\dot{p} = w \times q$ we get $k\dot{q} = w \times (kq)$, or $\dot{q} = w \times q$. Hence the equations are equivalent.

The condition that these lines shall intersect is $(p^{-1} \times \dot{p} - q^{-1} \times \dot{q}) \times p \cdot q = 0$, the terms in the parenthesis being the perpendiculars (vectors) from O to the lines. Expanding we get $\dot{p} \cdot q - p \cdot \dot{p} p^{-1} \cdot q + p \cdot \dot{q} - p \cdot q^{-1} \dot{q} \cdot q = 0$ which is satisfied by virtue of equations (1). Thus every pair of lines associated as above with a pair of points of the body will intersect. All of the lines will thus pass through the same point (since not all are coplanar) forming a bundle the vertex of which is the tip of the vector w which satisfies all of the equations. Since w is the same for all points at a given instant, we have thus proved that the body rotates instantaneously about an axis through O with the angular velocity w.

To determine w it is necessary to have two non-coincident lines given, *i. e.*, we must have the velocities of two points (not collinear with O) given to determine the velocities of all the other points. Since w satisfies $\dot{p} = w \times p$ and $\dot{q} = w \times q$ it follows that $w = \dot{p} \times \dot{q} / \dot{p} \cdot q = - \dot{p} \times \dot{q} / \dot{q} \cdot p$.

If $\dot{p} = 0$ and $\dot{q} \neq 0$ one of the lines lies along p and the point P is on w, and therefore on the axis of rotation. If $\dot{p} = 0$ and $\dot{q} = 0$ there is no motion since the lines intersect at the origin and $w = 0$.

If the velocities of the two points are coplanar with the vectors of those points, it will be seen easily that the vertex of the bundle will be a point on the normal to the plane through O, and that the lines for all points of this plane will form a pencil in a parallel plane. Since w is perpendicular to every line of this pencil, only one line is needed for its determination. Thus if $\dot{p} = w \times p$, we have $w = p^{-1} \times \dot{p}$. Hence in the case of motion of a rigid body (one point fixed) in a plane the velocity of one point will determine those of all the other points.

If now the fixed point be given a velocity we see that the velocities of all of the points of the body will be determined in space by the velocities of three non-collinear points, and in the plane by the velocities of two non-coincident points.

III. Relating to the Graph of $Y = f(X)$ for Complex Variables.

By E. L. Rees, University of Kentucky.

The following is to suggest a slight modification of the geometric representation of $Y = f(X)$, (X and Y both complex) given by Professor Frumveller in the November, 1917, number of the Monthly.

Let $X = x + iy$ and $Y = u + iv$. Letting u be represented by perpendiculars to the plane of X we get a surface which pictures the variation of the real part of Y. On this surface draw the contours for $v = v_1$, v_2, etc., the consecutive v's differing by a constant. These contours enable us to visualize the variation of v, so that we have pictured the variation of both u and v and, therefore, of Y, all on one surface.

Let the surface be cut by planes parallel to the X-plane, the planes of all of the consecutive pairs being the same distance apart. Project these curves of intersection and the contours on the X-plane. The curves into which these curves project will be the images of the grating of lines parallel to the u and v axes in the Y-plane. We thus establish a simple relation between this scheme and the usual scheme of representation of functional correspondence in complex variable theory.

For the linear function the surface is a plane and the contours are lines in this plane perpendicular to its xy-trace. If the coefficients are real the plane is parallel to the y-axis.

For the quadratic function the surface is a hyperbolic paraboloid and the contours are the intersections of this surface and hyperbolic cylinders.

For $Y = \log X$ the locus is a funnel-shaped surface generated by revolving $u = \log x$ about the u-axis and the contours are the meridians of this surface.

The projections of these contours and the intersections of the surfaces by the parallel planes on the X-plane give the familiar curves associated with these functions.

BUREAU OF INFORMATION.

James Byrnie Shaw, Chairman, University of Illinois, Urbana.

The Bureau of Information was established to furnish information to any one about any mathematical subject or question. So far as possible the committee will answer questions of the following character:

1. Definitions of terms.
2. Brief explanations of recent developments in mathematics.
3. Inquiries as to theorems.
4. Inquiries as to articles upon special topics.
5. Bibliographical references.
6. Historical questions.
7. Inquiries for book reviews.

8. Indication of libraries which will lend books.
9. Information as to graduate courses, fellowships, etc.
10. Inquiries as to societies, prizes, etc.

Whenever an inquiry seems to be of sufficient general interest to warrant publication, an article relative to it will appear in the MONTHLY.

The following question is of general interest and the reply is printed below:

"It is well-known that the equilateral triangle, the square and the hexagon, are the only regular figures that will perfectly fill a plane surface. What are the closed forms (polyhedra) that will completely fill space?"

REPLY BY ARNOLD EMCH, University of Illinois.

The cellular subdivision of space is one of the important problems of analysis situs and has been the subject of a great number of investigations. It comprises the theory of crystalline forms and structure, the regular division of the surface of a sphere, etc.

In this connection two representative treatises may be mentioned: *Mathematical crystallography*[1] by H. Hilton and *Vielecke und Vielflache. Theorie und Geschichte*[2] by Dr. M. Brückner, where certain aspects of the problems in question are thoroughly treated. In Brückner's book also numerous historical and other references may be found.

In the plane there are three cases of regular polygons by which a certain portion of the plane may be packed uniformly, namely, the triangle, the square, and the hexagon. In space of three dimensions there are equal cubes, belonging to the class of regular polyhedrons, by which a certain portion of space may be uniformly filled.

If we do not limit ourselves to regular polyhedrons then there are innumerable possibilities of subdividing a certain portion of space by a set of equal polyhedrons of semi-regular and irregular types. A few characteristic cases shall be given.

In the first place space may be subdivided in an infinite number of ways by rhombohedrons or parallelepipedons, whose faces are parallelograms which, in sets of two, are equal.

The subdivision itself is obtained as follows: Take three distinct concurrent, non-coplanar axes which are supposed to be indefinitely extended. Denote the axes by X, Y, Z, and the origin by O. From O measure off on these axes (on each axis with a definite unit-length) equal distances, so that on each axis the scale of real integers is represented. Through the division points on the X-axis draw planes parallel to the YZ-plane; through those of the Y-axis planes parallel to the ZX-plane; through those of the Z-axis planes parallel to the XY-plane. These three sets of planes divide space into a uniform set of equal parallelepipedons, whose vertices form a space-lattice. There is an infinite number of subdivisions possible where the faces of the parallelepipedon are equal rhombs. In this particular case the parallelepipedon is called a rhombohedron. Of the

[1] The Clarendon Press, Oxford, 1903.
[2] B. G. Teubner, Leipzig.

class of rhombohedrons one is of particular importance as will appear from the construction of the rhombus-dodecahedron, Figure 1, which is another example of a semi-regular polyhedron by whose repetition a certain portion of space may be uniformly packed.

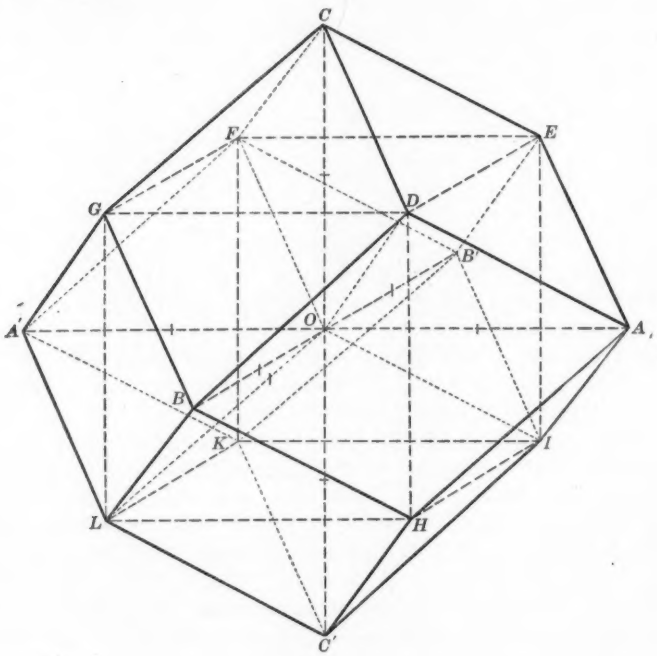

FIG. 1.

Draw three equal, mutually perpendicular axes AA', BB', CC', intersecting at O, and so that $OA = OA'$, $OB = OB'$, $OC = OC'$. Thus $AA'BB'CC'$ are the vertices of an octahedron. Now draw the perpendicular bisecting planes of the segments OA, OA', OB, OB', OC, OC'; they bound a cube $DEFGHIKL$. Join the vertices of the octahedron to those of the cube, as indicated in the figure. The joins thus obtained form the edges of the rhombus-dodecahedron, whose faces are equal rhombs with angles θ and ϕ, for which $\tan \theta/2 = \sqrt{2}$, or $\theta/2 = 54° 42' +$, and $\phi = 180 - \theta$. The planes of two adjacent rhombs include an angle of 120°. The rhombus-dodecahedron can be divided into the four equal rhombohedrons $ADOIHBLC'$, $DCFOBGA'L'$, $ADOIDCEB'$, $OIB'FLC'KA'$.

If we make an isometric projection of the figure, *i. e.*, if we assume in the figure OA, OB, OC at angular intervals of 120°, then OD which lies in one of the principal diagonals of the cube will merely appear as a point O, Figure 2. Likewise with the edges AI, HC', BL, GA', CF, EB', which are equal and parallel

to *DO*. The whole rhombus-dodecahedron appears as a regular hexagon with its radial lines, as represented by the internal hexagon around *O* in Figure 2. Without entering into further details, this shows that the rhombus-dodecahedron may be slipped into the space of a regular hexagonal prism. We can now add six other equal rhombus-dodecahedrons 1, 2, 3, 4, 5, 6, as shown in the figure, which fit in

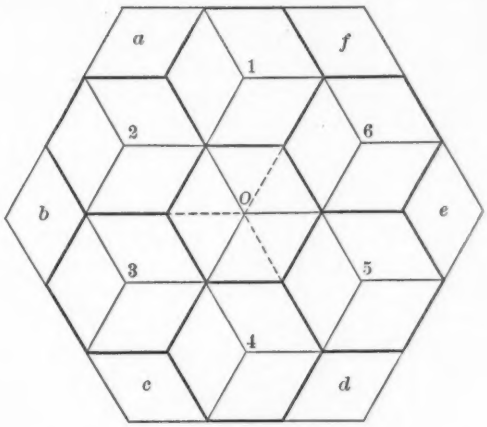

FIG. 2.

closely around *O*. Other dodecahedrons of the same kind may be fitted in horizontally and vertically, so as to pack uniformly a certain portion of space, which may be made as large as we please. If as shown in Figure 2, we fit in rhombohedrons *a, b, c, d, e, f* of the kind of which the dodecahedron is composed, *a regular hexagonal prismatic space may be packed uniformly by rhombus-dodecahedrons and rhombohedrons, or by rhombohedrons of the special form alone.*

As a last example of cellular division of space the well-known case of the honey-bees' cell may be mentioned. Geometrically the cell is obtained as follows: Take three faces of a cube adjacent at one of its vertices. Through the six "open" edges of the three faces pass planes parallel to the principal diagonal of the cube passing through the common vertex of the three faces. These six planes together with the three square faces of the cube enclose essentially the form of the bees' cell.

Further references on this extensive subject may be found in the treatises mentioned above.

UNDERGRADUATE MATHEMATICS CLUBS

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF MAINE, Orono, Me.

Scientific organizations at the University of Maine have gradually tended to become more special in their nature. The Maine State College, and later University of Maine, Scientific Association was probably organized in the seventies and flourished until some time in the nineties. In a sense The Mathematics and Physics Club was a successor to this organization and lasted for several years before disintegration by reason of the fact that "many of the men formerly interested in it were engineers, and several engineering clubs and societies were formed which occupied the attention of such men. The activities of this club were in part transferred for a time to voluntary meetings of the members of the mathematics department with students in that department, and of the physics faculty with major students in physics." But in February, 1916, it was felt that there was sufficient interest to maintain a separate organization and the new club was founded.

Its object is "to promote interest in the study of mathematics" and its members "consist of all persons connected with the departments of mathematics and astronomy and of any other persons sufficiently interested in mathematics to make application for a membership and who shall be elected by written ballot by a majority of the members present at any regular meeting." Average attendance this year about 8 of the 18 members.

Officers, 1917–18: President, Edith E. De Beck '18; vice-president, Samuel Guptill '20; secretary, Mildred Bisbee '20. The program committee consists of Professor M. O. Tripp, chairman, and the officers.

The following meetings have been held in 1917–18:

November 7: Business, including election of officers to fill vacancies caused by war conditions.

November 14: "Duplication of the Cube" by Samuel Guptill '20.

December 5: "Comparison of the Different Methods Used in the Duplication of the Cube" by Maynard Jordan and Quentin Stauffer, instructors in mathematics.

December 19: "History of Trigonometry" by Edith Deering '21.

January 10: "Mathematics for Engineering Students" by Professor Tripp; "History and Teaching of Mathematics in the United States" by Mildred Bisbee '20.

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill.

This club was organized in January, 1916, with a membership of 30. Membership was restricted to: (1) Members of the faculty, and graduate students,

in the departments of mathematics, physics, engineering and astronomy; (2) Seniors who are taking their major work in mathematics; (3) Students who have creditably completed 18 semester hours of work in the department of mathematics.

At a recent meeting, an amendment to the constitution was passed, extending membership to Juniors who are taking one or more courses as advanced as "C" in the department of mathematics. Representative courses of "C" grade are: higher algebra, solid analytic geometry, and advanced calculus.

Last year the Mathematics Club contained about 40 active members. At present (February 1) the active membership is about 20; but it is expected that the extension of membership privileges to Juniors will considerably increase the number.

Under the membership requirements the officers of the club are necessarily seniors or graduate students. Officers, 1917–18: President, Theodore Doll '17; vice-president, Franklin Mohr '18; secretary, Mae Campbell '18; treasurer, Helen Maloney '18; faculty adviser, Professor Robert E. Wilson. The program committee is composed of the faculty adviser, president, vice-president, and two members chosen by these three. The other two members are: Anastasia Zachariou '18 and Elizabeth Sheldon '18.

Meetings of the club are now held once in three weeks. Last year they were held once in two weeks. The lectures are about evenly divided between students and faculty. The programs for 1916 were as follows:

February 16: "The Construction of a Conic, given five Elements two of which are Imaginary" by Professor Thomas F. Holgate.
March 1: "What is Mathematics?" by Theodore Doll '17.
March 16: "Computing Machines" by Henry A. Babcock Gr., Fellow in physics.
March 29: "The History of π" by Minna Schick '16.
April 12: "The Method of Least Squares" by Harvey A. Anderson Gr.
April 26: "The Algebra of Al-Khowarizmi" by Ruby Lane '16; "The Hindu Arabic Numerals" by Harriet Knudsen '16.
May 10: "Descartes's Rule of Signs" by Professor David R. Curtiss.
May 24: "The Poincaré World" by Elizabeth Soderholm Gr.
October 12: "The Determination of Stellar Parallax" by Professor Philip Fox, director of Dearborn Observatory.
October 26: "Poincaré's Science and Hypothesis" by Dorothy Andrews '17.
November 9: "Arithmetical Stunts" by Leonard Janes '17.
November 23: "Euler" by Mabel Dinsmore '17; "Laplace" by Laura Hill '17.
December 7: "Railroad Curves" by Professor William H. Burger.

The following programs were given in 1917 and in the early part of 1918:

January 11: "Moritz's Memorabilia Mathematica" by Dorothy Waugh '17, Ruth Austin '17 and Robert Smyth Gr.
January 25: "Mathematical Journals" by Professor Curtiss.
February 15: "Galileo" by Henry Crew, professor of physics.
March 1: "Paper Folding" by Olga Podlesak '17.

March 15: "Circular Diagrams of Stress and Strain, and the Ellipse of Elasticity" by Hugh P. Bersie '17.

March 29: "Triangles and Squares" by Frank E. Wood, instructor in mathematics.

April 12: "Newton as Astronomer" by Luella Sayer Gr.; "Newton the Mathematician" by Helen Forbes '17; "Newton-The Leibnitz Controversy" by Ella Schneider '17; "Newton the Physicist" by William H. Grau, assistant in physics.

April 26: "History of Trigonometry" by Mary K. Voorhees '17; "Fourth Dimensional Geometry" by Frederick L. Kerr, Jr.

May 10: "Line Coordinates" by Professor Chester H. Yeaton.

May 24: "Some Generalizations of Rolle's Theorem" by Minna Schick, Jr.

November 15: "The real Roots of a Cubic Equation" by Professor Wilson.

December 5: "The Trigonometric Solution of the Cubic" by Anastasia Zachariou '18.

January 10: "The Problem of Two Bodies" by Franklin Mohr '18.

January 24: "The Mathematics of Field Artillery" by Professor E. J. Moulton.

February 14: "The Use of Complex Numbers in Electrical Theory" by William C. Bauer, professor of electrical engineering.

A service flag for the Mathematics Club would contain six stars. For all the information given above concerning the Club the editor is greatly indebted to President Doll who expects to be called to the colors before June.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon.

This club was organized in October, 1916, "to promote a broad interest in the whole field of mathematics, especially in phases of it that are not dealt with in the regular courses; also to touch upon interesting and useful applications, the human element and the recreational." All students in the department of mathematics and allied departments are eligible for membership. There are 30 members; average attendence about 20.

Officers, 1917–18: President, Olga Soderstrom '18; vice-president, Ruth Westfall '18; secretary, Helen Wells '18; treasurer, Kenneth Moores '18. The program for each meeting is prepared by the officers assisted by the instructors in the department.

The program for the year 1916–17 follows:

October 6: "Early Reminiscences of the University of Oregon" by Edward H. McAlister, professor of mechanics and astronomy; "The History of Johns Hopkins University" by Roy M. Winger, professor of mathematics.

November 1: "The Fallacies of Arithmetic" by Lucile Watson '17; "The Construction, Theory, and Purpose of the Magic Square" by Loren Roberts '17.

December 6: "The Construction of Polygons by the Use of Ruler and Compasses" by Kenneth Moores '18; "The Problem of Squaring the Circle" by Ruth A. Westfall '18.

March 7: "Reminiscences of the Universities of Wisconsin, Chicago, and Yale" by Edgar E. De Cou, head of the department of mathematics; "Mathematical Recreations, Puzzles and Catch-problems" by Olga Soderstrom '18.

May 23: "The History of Correspondence Schools and the Progress of our University in this Regard" by Chester Kronenberg, instructor in mathematics; "The Application of Mathematics to Wave Surfaces" by Frederick E. Melzer '17.

The programs for the first two meetings of 1917–18 are as follows:

December 13: "The Problem of the Duplication of the Cube" by Elizabeth Carson '18; "Mathematics and the War" by Professor De Cou.

January 16: "The Problem of the Trisection of an Angle" by Cornelia Heess '18; "The Newton-Leibnitz Controversy over Calculus" by Edward E. Bentley '20.

THE MATHEMATICAL AND ASTRONOMICAL CLUB OF SWARTHMORE COLLEGE, Swarthmore, Pa.

This club was founded, in March, 1907, in order "to give students and faculty an opportunity to report upon subjects not usually treated in the classroom. It affords also an opportunity to bring to the students of mathematics from those outside the college two or more addresses which are given each year before the club; but the chief object is to develop in students the responsibility for certain exercises." Anyone desiring to become a member may do so. There are 42 members this year and the average attendance is about 30. No regular club dues are levied, but each year an assessment is made for current expenses; it never exceeds a dollar and is usually less than half that amount.

The following programs were given in the early part of 1917–18:

October 2: "Properties of Sound" by Harvey C. Hayes, professor of physics.

October 16: "Popular Phases of the Fourth Dimension" by Ethelwyn Bower '18.

November 6: "Determination of Latitude and Longitude by Means of Celestial Objects" by John Trimmer '18; "The Use of the Sextant" by Harry Yardley '19; "The Log Book of a Ship" by Ewing Carson '18.

November 20: "Opportunities open to Students of Mathematics" by John A. Miller, head of the department of mathematics and astronomy; "Plain Sailing," by Frank Fetter '20; "Mercator's Projection" by Robert Blau '18.

December 4: "Algol and its Use in the Determination of Light Wave-Lengths" by Dorothy Johnson '18; "History of the Parallax Problem" by Caroline Smedley Gr.; "Use of the Telescope and Galvanometer" by Hazel Brown Gr.; "Researches of Sproul Observatory" by Professors Miller, John H. Pitman (mathematics and astronomy), and Hayes.

December 18: "The Depression Range Finder and its Use in the Present War" by John Trimmer '18.

THE MATHEMATICS CLUB OF VASSAR COLLEGE, Poughkeepsie, N. Y.

This club was organized, in January, 1916, "to bring together students interested in mathematics, and to give them an opportunity to present papers themselves and to hear and discuss those presented by their fellow students. All students who have completed two semestral courses of elective[1] mathematics, and members of the faculty in the mathematics department are eligible for active membership. Students who have taken or are taking one course in elective mathematics, and members of the faculty in other than the mathematics department are eligible for associate membership." The club has 37 active members practically all of whom, and others, are present at each of the monthly meetings.

Officers, 1917–18: President, Beatrice Boyden '18; vice-president, Rachel G. Franklin '19; secretary-treasurer, Louise E. C. Stuerm '19; faculty member of executive committee, Professor Elizabeth B. Cowley; member at large on executive committee, Ruth De Land '18.

The following are programs for 1917–18:

October 30: "The Parallel Curves of the Conics" by Dr. Mary E. Wells.

November 13: "The Mathematics of Warfare" by J. Malcolm Bird, associate editor of *The Scientific American*.[2]

December 13: "The Slide Rule" by Helen Thompson '18; "Magic Squares" by Louise E. C. Stuerm '18.

January: "Planimeters" by Marjorie Wheatley '18; this paper was illustrated by planimeters made by the speaker's father, who is an engineer.

TOPICS FOR CLUB PROGRAMS.

6. WOMEN AS MATHEMATICIANS AND ASTRONOMERS.

The first woman mathematician regarding whom we have positive knowledge is the celebrated Hypatia,[3] recognized head of the Neoplatonic school, who was barbarously murdered at the hands of an Alexandrian mob in the early part of the fifth century. She was the daughter of Theon, also a mathematician and philosopher, who edited Euclid's *Elements*.[4] "Her great eloquence and rare modesty and beauty combined with her remarkable intellectual gifts attracted to her class-room a large number of pupils." She is said to have been the author of commentaries on works by Diophantus, Apollonius of Perga, and Ptolemy.

Maria Gaetana Agnesi, an Italian mathematician, linguist and philosopher of the eighteenth century was one of the twenty-three children of a rich and

[1] Vassar College requires a year of mathematics for freshmen.

[2] Mr. Bird's lecture was open to all members of the College and was attended by about 500 people, including practically all of the students of mathematics, and others. In *The Mathematics Teacher*, September, 1917, vol. 10, pp. 35–51, appeared an article on "The Mathematics of Warfare" by Mr. Byrd.

[3] See *Encyclopædia Britannica*, eleventh edition, 1910, article "Hypatia" with bibliography. Kingsley's romance will be recalled.

[4] *Cf.* HEATH, *The Thirteen Books of Euclid's Elements*, vol. 1, 1908, Chapter 5.

cultured gentleman of Milan who had a taste for science. "When only nine years old she had such command of Latin as to be able to publish an elaborate address in that language, maintaining that the pursuit of liberal studies was not improper for her sex. By the thirteenth year she had acquired Greek, Hebrew, French, Spanish, German and other languages. Two years later her father began to assemble in his house at stated intervals a circle of the most learned men in Bologna,[1] before whom she read and maintained a series of theses on the most abstruse philosophical questions" (*Encyclopædia Britannica*). Her father caused records of these meetings to be published when she was twenty years old. Then followed ten years of arduous labor, entered upon for the benefit of one of her brothers who had a taste for mathematics, and resulting in the publication of *Le Instituzioni Analitiche* (two quarto volumes, Milano, 1748). It is here (Vol. 1, pp. 380–81) that we find a discussion of the curve commonly known now as the "witch of Agnesi."[2]—French and English translations of this work were published in 1775 and 1801[3] respectively. In 1750 she was appointed professor of higher mathematics and natural philosophy[4] at the University of Bologna but never lectured. For the rest of her long life—she lived to be 81 years of age—she was occupied in charitable work and religious meditations. A younger sister was a well-known pianist and composer of three operas.

Among French representatives of women as mathematicians Sophie Germain (1776–1831) is the most notable. She was a pupil of Lagrange and a coworker with Biot, Legendre, Poisson and Lagrange. In 1816 the Institute of France awarded a prize for her memoir on the mathematical theory of the vibrations of elastic surfaces and its comparison with experience. This and other memoirs have been reviewed by Todhunter.[5] Her fame brought her into contact with such men as Ampère, Cauchy and Gauss.[6] Terquem has written an interesting biographical sketch.[7]

Russia produced Sonja Kovalevski (1850–1891) who studied mathematics for two years at the university of Heidelberg (under H. v. Helmholtz, Kirchhoff, Königsberger and du Bois-Reymond) and then proceeded to Berlin, where she

[1] Milan? Compare note 4.

[2] See F. G. Teixeira, *Traité des courbes spéciales remarquables planes et gauches*, Coïmbre, 1908–9, Vol. 1, pp. 108–115, 385–6; Vol. 2, pp. 405–6.

[3] The review of this edition in the *Edinburgh Review* for January, 1804, is interesting.

[4] There is no basis in fact for the statements of the *Encyclopædia Britannica* to the effect that (1) Agnesi's father was professor in the University of Bologna or (2) her appointment as professor had anything to do with his illness. *Cf.* the best biography, L. Anzoletti, *Maria Gaetana Agnesi*, Milano, 1900, pp. 273ff. There is a sketch in *The Catholic Cyclopedia*, New York, Appleton, Vol. 1, 1907.

[5] *History of the Theory of Elasticity and of the Strength of Materials*, Vol. 1, Cambridge, 1886, pp. 147–160; see also Chasles, *Rapport sur les progrès de la géométrie*, Paris, 1870, pp. 35–36; and H. Stupuy, *Œuvres philosophiques de Sophie Germain, suivies de pensées et de lettres inédites et précédées d'une étude sur sa vie et ses Œuvres*, Paris, Quantin, 1879; another edition, 1896.

[6] *Cf.* Boncompagni, *Lettera inedita di Carolo Fredrico Gauss a Sofia Germain*, Firenze, 1871; reviewed by C. Henry in *Bulletin des sciences mathématiques* (Darboux), Paris, tome 14, 1879, pp. 401–406.

[7] *Bulletin de bibliographie, d'histoire et de biographie mathématiques*, tome 6 (1860), pp. 9–12 (*Nouvelles Annales de mathématiques*, tome 19).

prevailed upon Weierstrass to give her private lessons, since the doors of the university were closed to women. He soon found that she had "the gift of intuitive genius to a degree he had seldom found among men, even his older and more developed students." At the end of three years she presented to the university of Göttingen three theses worked out under the direction of Weierstrass through whose strong recommendation she was awarded the degree of doctor of philosophy without examination, and in absentia. Not long after this (1874) she returned to Russia. In 1883 she was appointed lecturer in mathematics at the university of Stockholm, and in 1884 full professor. This post she held till her death. Five of her memoirs were published in *Acta Mathematica* and the most notable is that of 1889, "Sur le problème de la rotation d'un corps solide autour d'un point fixe," which, in 1888, won the prix Bordin (doubled in value because of the remarkable merit of the memoir) of the Institute of France. Sonja Kovalevski "died just as she had attained the height of her fame and had won recognition even in her own country by election to membership of the St. Petersburg Academy of Science." [1]

Of scientists in England a few notes may be given concerning the astronomer Caroline Herschel, who was born at Hanover, in 1750 and started her career in England, with her brother William, as teacher of music in Bath. When her brother accepted the office of astronomer to George III, she became his constant assistant in his observations and also executed laborious calculations which were connected with them. She discovered comets and remarkable nebulæ and completed the reduction to January, 1800, of 2,500 nebulæ discovered by her brother. The Royal Astronomical Society awarded her its gold medal and elected her an honorary member. She died at the age of 98, having spent about fifty years of her chief scientific activity in England.[2]

Perhaps the best single work containing biographies of scores of women mathematicians and astronomers is the second edition of A. REBIÈRE, *Les femmes dans la science*.[3] The bibliographies are full and valuable. Among portraits, it contains those of Agnesi, Sophie Germain, Caroline Herschel, Kovalevski, Agnes Mary Clarke, the English author of various works on the history of astronomy, Marquise du Châtelet[4] (pupil of Clairaut, Maupertius and Jean Bernoulli, and translator of Newton's *Principia*), Martha Mitchell, the astron-

[1] For sketches of Kovalevski's life reference may be given to A. C. LEFFLER, *Sonya Kovalevsky*, biography and autobiography, translated into English, London (1) Walter Scott, 1895, (2) T. Fisher Unwin, 1895; to the article in the eleventh edition of the *Encyclopædia Britannica*, 1911; to G. MITTAG-LEFFLER, "Sophie Kovalevsky, notice biographique," *Acta Mathematica*, Vol. 16 (portrait frontispiece), pp. 385–392, 1893; and to E. W. CARTER, "Sophie Kovalevsky," *Fortnightly Review*, Vol. 63, 1895, pp. 767–783. For other references see *Bibliotheca Mathematica*, 3. Reihe, Band 2, 1901, p. 337.

[2] For detailed biographical material see Mrs. John Herschel's *Memoir and Correspondence of Caroline Herschel*, 2d edition, London, Murray, 1879; *Dictionary of National Biography*, London, Vol. 26, 1891; and the Introduction to Volume I of *The Scientific Papers of Sir William Herschel*, London, Royal Society and Royal Astronomical Society, 1912.

[3] Paris, Novy, 1897, 361 pp.

[4] For other portraits see Carlyle's Works, centenary edition, New York, Scribner, 1908, Vol. 14, p. 191; and G. C. WILLIAMSON, *History of Portrait Miniatures*, London, Bell, 1904, Vol. 2, plate 87.

omer of Vassar, Charlotte A. Scott, the mathematician at Bryn Mawr, Christine Ladd-Franklin of New York, and Dorothy Klumpe, a native of San Francisco, who is the only American who ever received the state doctorate in mathematical sciences in France.[1]

A more recent work by J. A. Zahm was published under the pseudonym H. J. Mozans with the title *Woman in Science with an introductory Chapter on Woman's long Struggle for Things of the Mind.*[2] While the book presents material of value for our topic it contains also many unreliable statements.

The highly interesting article by Gino Loria on "Les femmes mathématiciennes"[3] does not enter so much into details of the lives of the different women as into the consideration of the value of their contributions to science[4] in comparison with those made by many men. On the basis of available facts he finds himself obliged "with regret to make reservations in regard to those whom nature seems to have called to other destinies." Much the same conclusion is reached by the doctor, P. J. Möbius (grandson of A. J. Moebius) in his valuable *Ueber die Anlage zur Mathematik.*[5]

7. The Binary Scale of Notation, a Russian Peasant Method of Multiplication, the Game of Nim and Cardan's Rings.

The binary system is built upon a scale of 2 instead of 10, thus using only two figures, 1 and 0. The succession of integers represented in the decimal notation by

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \cdots,$$

would appear in the binary system as follows:

$$1\ \ 10\ \ 11\ \ 100\ \ 101\ \ 110\ \ 111\ \ 1000\ \ 1001\ \ 1010\ \cdots.$$

By this means all addition is reduced to counting, all multiplication to simple additions, and division to simple subtractions. One inconvenience of the binary system is the large number of figures which may be required for a relatively small number. The binary representation requires for large numbers approximately 10/3 of the number of figures in the decimal representation. Legendre used binary numeration for calculating high powers and gave the following note[6] on a "very short method for expressing a somewhat large number in binary scale. Let the number be 11,183,445; . . . divide by 64, then the remainder is 21 and the quotient 174,741, which, divided by 64, gives the remainder 21 and

[1] For a time, at least, Miss Klumpe was a member of the staff of the Paris Observatory.

[2] New York, Appleton, 1913. Chapter 3, pp. 136–166: "Women in Mathematics," chapter 4, pp. 167–196: "Women in Astronomy."

[3] *Revue scientifique*, Paris, Dumoulin, 1903, 4e sèrie, tome 20, pp. 386–392. See also G. Loria, "Donne matematiche," *Memoria letta il 28 dicembre*, 1901, *Atti e memorie della R. Accademia Virgiliana*, Mantova, 1903, pp. 75–98.

[4] In Note 1 of his Work ("Si la femme est capable de science," pages 287–319) Rebière considers this question. He quotes those holding opposing views, such as Molière, Prudhom, Schopenhauer, and Lalande, Stuart Mill, Anatole France.

[5] Leipzig, Barth, 1900. "Beiträge zur Kenntniss des Mathematischen Talentes. C. Ueber die mathematischen Weiber," pp. 77–86; portraits of Germain, Herschel and Kovalevski.

[6] A. M. Legendre, *Essai sur la théorie des nombres*, Paris, 1798, p. 229; 2e édition, 1808, p. 209.

the quotient 2730; finally 2730 divided by 64 gives the remainder 42 and the quotient 42; but 21 is expressed in binary numbers by 10101 and 42 by 101010. Then the proposed number is expressed by 101010 101010 010101 010101."

Fractions, decimals, subtraction, division, etc., in the binary system have been discussed by E. Collignon.[1] Reference may be given also to articles by C. Berdellé,[2] to the sketch by É. Lucas in *Récréations mathématiques*,[3] to an extract from the first lesson of Laplace at the *École Normale*, "le 1er pluviôse an III" [1795],[4] and to a note by E. Miot.[5]

Leibniz is often referred to as the founder of binary arithmetic. Whether this be true or not it is certain that the subject interested him considerably and that his publications[6] influenced, directly or indirectly, the production of a series of works such as those by T. F. De Lagny,[7] J. B. Wiedeburg,[8] F. S. Brunetti,[9] and G. F. Brander.[10] As early as 1698 Leibniz set forth the foundations of his binary arithmetic in a letter to Schulenburg and the matter occupied his attention for twenty years. His first published paper on the subject was in 1703. As its title indicates Leibniz believed that this arithmetic solved the mystery of a Chinese symbol Je-Kim (book of mutations) attributed to Fo-hy, the most ancient legislator in China. The symbol is composed of 64 figures each formed of six horizontal lines (some whole, others broken in the middle)one above the other.[11] Interpreting each whole line as corresponding to 1 and each broken line to 0, Leibniz translated the figures as corresponding to 0, 1, 2, $\cdots$, 63. But such an interpretation has been shown to be worthless.[12] In honor of Leibniz's discoveries in connection with binary arithmetic a medal was struck by Rudolph August,

[1] "Note sur l'arithmétique binaire," *Journal de mathématiques élémentaires* (De Longchamps), 1897, tome 21, pp. 101–106, 126–131, 148–151, 171–174.

[2] *Association française pour l'avancement des sciences:* (*a*) "La numération binaire et la numération octovale," 1887; (*b*) "Arithmétique de la gamme," 1897; (*c*) "Les curiosités du calcul," 1898. Also *L'Intermédiaire des mathématiciens*, 1904, tome 11, pp. 269–271.

[3] 2e édition, Paris, Gauthier-Villars, 1891, pp. 143–160.

[4] *L'Education mathématique*, 7e année, 1904, pp. 59–60.

[5] *L'Intermédiaire des mathématiciens*, 1911, tome 18, p. 190.

[6] (*a*) "Explication de l'arithmétique binaire, qui se sert des seuls caractères 0 et 1: avec des remarques sur son utilité et sur ce qu'elle donne le sens des anciennes figures Chinoises de Fohy," *Histoire de l'académie des sciences de Paris*, 1703, pp. 58–63; *Mémoires*, pp. 85–89. Also, *Opera omnia*, Genevæ, vol. 3, 1768, pp. 390–394. (*b*) "De inventione arithmeticæ binariæ a G. G. Leibnitio. Excerpta ex vitâ Leibnitii a D. Jancourt scripta," *Opera omnia*, vol. 3, pp. 346–348. (*c*) "G. G. Leibnitii epistolæ duæ ad J. C. Schulenburgium, De Arithmeticâ Dyadicâ . . .," *Opera omnia*, vol. 3, pp. 349–354.

[7] *L'arithmétique nouvelle*, Rochefort, 1703. (Hutton states that De Lagny proposed a new system of logarithms, on the plan of binary arithmetic, which he found shorter and more easy and natural than the common ones.)

[8] *Dissertatio mathematica de praestantia arithmeticæ binariæ prædecimali . . .*, Jenae, 1718.

[9] *Arithmetica binomica e diadica, in cui tutte le operazioni si fanno colle sole figure Uno, e Zero* (Riflessione al Giornale de'Letterati di Roma an 1746, page 884). Roma, 1746; another edition, 1754.

[10] *Arithmetica binaria sive dyadica, das ist, Die Kunst nur mit zwey Zahlen in allen vorkommenden Fällen sicher und leicht zu rechnen*, Augsburg, 1769; another edition, 1775.

[11] *Cf.* P. Carus, "Chinese Philosophy," *The Monist*, Chicago, 1896, vol. 6, especially pages 191 ff.

[12] *Annales du musée guimet*, tome 8 (1885) and 23 (1893).

duke of Braunschweig. A copy of this medal forms the frontispiece to Wiedeburg's dissertation and is also given on p. 1002 of *Collectanea de Breslau.*

The problem of calculating π in the binary system, proposed several times by Leibniz (*Opera omnia*, tome 3, pp. 521, 547···), was solved by Jacob Bernoulli.[1] For $\log \pi$ in binary system see *Journal de Mathématiques élémentaires* (Bourget), Paris, tome 2, 1878, p. 229.

Peano has indicated applications of the binary system in "La numerazione binaria applicata alla stenografia," *Atti della R. Accademia delle Scienze di Torino*, 1898; (*b*) *Formulaire de Mathématiques*, 1901, pp. 77, 154, 177.

The source of the many references to a Russian peasant method of multiplication seems to have been the publication in 1896 by G. De Longchamps of a request from a Mr. Plackowo (of Tokarewka in the government of Tamboff, Russia) for an explanation of the method. An example will suffice to illustrate both the method and the connection with the binary system of numeration. Suppose, for example, the problem is to multiply 35 by 42. Divide one of the numbers (say 35) by 2, giving 17 (remainder neglected) and multiply 42 by 2; continuing the process we get the following series of corresponding numbers:

35	17	8	4	2	1
42	84	168	336	672	1344.

Add the numbers of the second row corresponding to odd numbers in the first row and we obtain the required product 1470. In *Mathematical Gazette*, 1917, Vol. 9, p. 9, G. H. Bryan has a note on "'Russian Peasant' Multiplication in Roman Numerals." References may be given also to: J. BOWEN, "The Russian Peasant Method of Multiplication," *The Mathematics Teacher*, 1912, Vol. 5, pp. 4–8; E. CZUBER, "Ueber ein Multiplikationsverfahren," *Zeitschrift für das Realschulwesen*, Dezember, 1915.

The application of the binary scale of notation to "Nim, a Game with a complete Mathematical Theory" was explained by C. L. BOUTON in *Annals of Mathematics*, 1901, 2d series, Vol. 3, pp. 35–39. To him is due also the name which was chosen in preference to Fan-tan, which had been used, because it is not the same as the Chinese game of that name. "The game is played by two players. Upon the table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no piles shall be equal at the beginning. A play is made as follows: The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins." See also E. H. MOORE, "A Generalization of the Game called Nim," *Annals of Mathematics*, 1910, 2d series, Vol. 11, pp. 93–4.

The toy usually known as "Chinese Rings"[2] seems to have been first de-

[1] *Cf.* LEIBNIZ, *Math. Schriften* (Gehrhardt), vol. 3, p. 97.

[2] In French: baguenaudier.

scribed by CARDAN on page 294 of his *De Subtilitate*, Nuremberg, 1550. It is also discussed by John Wallis in the Latin edition of his *Algebra*, 1693 (*Opera*, Vol. 2, pp. 472–478). The toy "consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed." A pretty full discussion of the number of operations required to remove the rings from the bar (a problem which Cardan and Wallis failed to solve) has been given in W. W. R. BALL, *Mathematical Recreations and Essays*, 5th ed., London, Macmillan, 1911, pp. 230–234. Practically all that is in Ball's sketch, and considerable more, is given in E. LUCAS, *Récréations Mathématiques*, Paris, Gauthier-Villars, tome 1, 2e éd., 1891, pp. 161–186. The discussion of the number of operations by means of the binary scale is due to [L. GROS], *Théorie du Baguenodier*, par un clerc de notaire lyonnais, Lyon, 1872.

NOTES AND NEWS.

Professor E. W. BROWN, of Yale University, was elected a member of the Council of the American Philosophical Society at the January meeting.

The Paris Academy of Science during the year 1917 awarded seventy-seven prizes for especial achievements in science, eight of which were granted to mathematicians.

The Royal Society held its regular meeting on December 6, 1917, under the presidency of Sir J. J. THOMPSON. The only mathematical papers presented were "The series of Legendre," by W. H. YOUNG, and "The Zeros of Bessel functions," by G. N. WATSON. The London Mathematical Society held its monthly meeting on the same date, with vice-president H. HILTON presiding. The following papers were presented: "A new method of describing three-bar curves," by R. L. HIPPISLEY; "Proof of the primality of $N = (10^{19} - 1)/9$," by O. HOPPE; "New Tauberian theorems," by HARDY and LITTLEWOOD; "The curves which lie on the quartic surface in space of four dimensions," by C. V. H. RAO; "The connection between Legendre series and Fourier series," and "Series of Bessel functions," by W. H. YOUNG.

The third volume of the *Proceedings* of the National Academy of Sciences of the United States of America has been completed. The membership of the Academy now numbers one hundred fifty-eight, distributed somewhat irregularly over the United States, and representing the fields of mathematical, physical and biological sciences. Twelve members of the Academy are now, or have been, professors of mathematics. The table of contents of papers presented shows thirteen titles belonging to the field of mathematics, only two of which were credited to members of the Academy. The roster also shows twenty-six foreign associate members, four of whom are noted mathematicians, two belonging to the central, and two to the allied powers.

The Department of Superintendence of the National Educational Association met in Atlantic City on February 26–March 2, 1918. The program was rich in variety and extent, especially in respect to all phases of activity which in any way pertained to the world war. One feature of the program may well be commended to the mathematical fraternity, namely, the two sessions devoted to the teaching of English under the auspices of the National Council of Teachers of English. It has been suggested that our Association could do good service by providing speakers for the program of educational meetings both state and national, and it would seem that this Department of Superintendence offers a most favorable opportunity for propaganda of this kind.

The series of articles on "Valid aims and purpose for the study of mathematics in secondary schools" which is now running in *School Science and Mathematics* is commended to the attention of teachers of mathematics of all grades. These articles are the outcome of the work of a committee of the Mathematics Club of Chicago. Reprints of the whole series may be obtained from the chairman of the committee, Mr. ALFRED DAVIS, of the Francis W. Parker School, 330 Webster Avenue, Chicago.

Two articles bearing on the attacks upon mathematics are printed in *School Science and Mathematics* for January and February, 1918. They are by Professor J. W. A. YOUNG, of the University of Chicago, and relate to the psychological investigations in the disciplinary value of studies. The first article gives a summary of the work which has been done by psychologists along this line with a bibliography and definite references; the second gives some of the author's own theory on the subject, especially as related to mathematics. The psychological discussion of the value of studies is also presented in an interesting address by Professor E. C. MOORE before the Association of Mathematics Teachers of New England, and published in full in *School and Society*, October 27, 1917.

Ten men connected with the department of mathematics at the University of Illinois have resigned to enter the service of the government since the United States entered the war. The following five resignations have not been noted in the MONTHLY: Dr. L. M. KELLS, instructor in mathematics, has entered the Officers' Reserve Training Camp at Battle Creek, Michigan; Dr. J. R. MUSSELMAN, instructor in mathematics, has begun statistical work for the government at Washington; Assistant H. D. FRARY has been appointed director of the wood-testing plant for aëroplanes, at the University of Wisconsin; Assistant W. E. EDINGTON has accepted a position in the research division of the Signal Service and is located at Leavenworth, Kansas; and Assistant A. W. LARSEN has resigned to accept an instructorship in mathematics at the University of Kansas. Two of these vacancies have been filled by the appointment of Dr. J. E. MCATEE, of William Jewell College, as instructor in mathematics, and the appointment of Mr. L. L. STEIMLEY, former instructor at the University of Kansas, as assistant in mathematics.

If departments of mathematics of our colleges and universities, offering summer courses in mathematics, will report to the editor of Notes and News an outline of these courses at once, we shall undertake to insert the announcements in the May issue of the MONTHLY. The outline should be prepared in the form used in the MONTHLY for April, 1917. Last year a fairly complete synopsis of summer work offered at the various institutions was secured by sending personal letters to each institution; no such letters will be sent this year, but it is hoped that all departments offering summer work will send promptly the synopsis for insertion in the MONTHLY.

The University of California will this year conduct two summer sessions, June 24 to August 3, one at the University of Berkeley, the other in Los Angeles. The department of mathematics will offer at Berkeley courses in elementary mathematics, advanced algebra, calculus, differential equations, theory of numbers, theory of functions, and integral equations. The instructors will be Professor D. N. LEHMER, Instructor BERNSTEIN and two assistants, and Dr. G. C. EVANS, of Rice Institute. At Los Angeles Professor C. A. NOBLE, and Professor V. SNYDER, of Cornell University, will offer courses in elementary algebra, plane analytic geometry, calculus, and higher geometry.

The April meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 12, 13. Besides the usual research papers on both days, there will be a Symposium on Summable Series on Friday afternoon. Professor R. D. CARMICHAEL, of the University of Illinois, will present a paper on "General aspects of the theory of summable series," and Professor C. N. MOORE, of the University of Cincinnati, will treat "Applications of the theory of summability to developments in orthogonal functions."

At the coming summer meeting of the ASSOCIATION a session is to be arranged for papers to be presented by members. The Program Committee requests that members who are ready to discuss questions of interest in connection with collegiate instruction of mathematics send abstracts of their proposed papers to the chairman, Professor R. C. ARCHIBALD, Brown University, as soon as possible, and not later than May 15. Such papers should, in general, be of such length as may be delivered in fifteen to twenty minutes. It is hoped there will be a large response to this appeal, in the interests of a live and varied program.

4

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXV APRIL, 1918 NUMBER 4

DESCRIPTIVE GEOMETRY AND ITS MERITS AS A COLLEGIATE AS WELL AS AN ENGINEERING SUBJECT.[1]

By WILLIAM H. ROEVER.

1. The Evolution of Descriptive Geometry. To the needs of the architect on the one hand and to those of the artist on the other, can be attributed the origin and much of the development of that branch of mathematics which is now called descriptive geometry. With the development of the builder's art, there must also have developed the process of making plans and elevations of buildings. In fact, the Roman architect Vitruvius used the terms "ichnographia" and "orthographia" for plan and elevation, respectively. In the attempts to solve the problems encountered in the construction of arches and other architectural forms, the art of stereotomy (stone and wood cutting) was gradually developed in the Middle Ages. To solve these problems the objects under consideration were represented by their plans and elevations, the usefulness of which was thus extended from that of mere representation to that also of the solution of stereometric problems. A remarkable step was taken in 1738 when Frézier separated the geometric constructions of this art from their technical applications and used them to solve problems of space. But it was Gaspard Monge (1746–1818) who, by enriching and systematizing this new constructive geometry of space, elevated it to the dignity of a pure science and gave to it the name descriptive geometry. Monge also solved with it the ordinary problems of geodesy and topography.

The needs of the artist, as has already been implied, were responsible for the early development of perspective. Exhumed mural paintings from ancient

[1] An address delivered at the third annual meeting of the Mathematical Association of America, held at the University of Chicago, December 27–28, 1917.